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Local Spline Approximation Methods*

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1. INTRODUCTION

The purpose of this paper is to construct some explicit polynomial spline approximation operators for real-valued functions defined on intervals or on reasonable sets in higher dimensions. Specifically, we consider operators of the form $Qf = \sum \lambda_i f N_i$, where $\{N_i\}$ is a sequence of B-splines and $\{\lambda_i\}$ is a sequence of linear functionals chosen so that

(i) Qf can be applied to a wide class of functions, including, for example, continuous or integrable functions;

(ii) Q is local in the sense that $Qf(x)$ depends only on values of f in a small neighborhood of x;

(iii) Qf approximates smooth functions f with an order of accuracy comparable to best spline approximation.

Such approximation schemes have several important advantages over spline interpolation. They can be constructed directly without matrix inversion, local error bounds are obtained naturally, and for lower derivatives the error bounds can be made independent of any mesh ratios.

Since the key to obtaining operators Q with property (iii) is to require Q to reproduce appropriate classes of polynomials, we begin by examining (in Section 3) when this is possible. This leads to a scheme for constructing

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λ_i as linear combinations of prescribed $\{\lambda_{ij}\}_{j=1}^{l}$ to produce Q reproducing polynomials of degree $l - 1$. We then find explicit representations for the coefficients and use them to obtain error bounds.

We illustrate our construction with two explicit classes of B-spline approximation methods; the first based on λ_{ij} which involve point evaluation of f (and/or its derivatives), and the second based on λ_{ij} involving integrals of f against appropriate polynomials. The first class includes the variation-diminishing method of Schoenberg and Marsden (see [12]), the projectors of de Boor [4], and the quasi-interpolant of de Boor and Fix [6]. (The latter in turn was shown in [6] to include the "approximation by moments" method of Birkhoff [2]; see also de Boor [3]). Other direct spline approximation methods have been considered by Babuska [1], Fix, Nassif, and Strang [8, 15], Jerome [10], and Schultz [13]. Our discussion owes much to the paper of de Boor and Fix [6]. However, in comparison with their quasi-interpolant, our method based on point evaluations has the advantages that it can be constructed without using derivatives while producing the same error bounds as their quasi-interpolant, and with the same mesh-independence for the lower derivatives, (cf. the open question in [6, p. 36]).

In Section 7 we consider when our local approximation schemes become projectors, and in Sections 8–10 we study a multidimensional scheme based on point-evaluation functionals. This method can be applied to functions defined on nonrectangular regions Ω (without extending the function), and the corresponding error bounds hold throughout Ω.

2. Splines and B-Splines

In this section we introduce a class of polynomial spline functions defined on an interval $[a, b]$ and give a basis of B-splines for it.

Let $a = y_0 < y_1 < \cdots < y_n = b$ and a corresponding sequence of positive integers $\{d_i\}_1^{n-1}$ be given. We write π for the nondecreasing sequence $\{x_i\}_0^N$ obtained from $\{y_i\}_0^n$ by repeating y_i exactly d_i times (thus $N = \sum_{i=1}^{n-1} d_i + 1$). If k is an integer with $k \geqslant d_i$, $i = 1, 2,..., n - 1$, we define

$$\mathscr{S}_{k,\pi} = \{g : g\,|_{(y_i, y_{i+1})} \in \mathscr{P}_k\,, \qquad i = 0, 1,..., n - 1 \text{ and}$$

$$g^{(j)}(y_i+) = g^{(j)}(y_i-), \qquad j = 0, 1,..., k - d_j - 1;\, i = 1, 2,..., n - 1\},$$

where $\mathscr{P}_k$ is the set of polynomials of degree less than k. This is the familiar class of polynomial splines of order k (degree $k - 1$) with knots (of multiplicity d_i) at the points y_i, $i = 1, 2,..., n - 1$.

To define a basis for $\mathscr{S}_{k,\pi}$, let $\pi = \{x_i\}_0^N$ be extended to a nondecreasing sequence $\pi_e = \{x_i\}_{1-k}^{N+k-1}$ with $x_i < x_{i+k}$, $i = 1-k,\ldots, N-1$. With

$$G_k(t; x) = (t-x)_+^{k-1},$$

we define

$$N_{i,k}(x) = (x_{i+k} - x_i)[x_i, \ldots, x_{i+k}]\, G_k(\cdot, x), \tag{2.1}$$

$i = 1-k,\ldots, N-1$, where the symbol $[x_i, \ldots, x_{i+k}]$ denotes the kth-order divided-difference functional. The $N_{i,k}$ are, apart from a constant factor, the B-splines of Curry and Schoenberg [7]. In the following lemma we summarize, for ready reference, several of their properties.

LEMMA 2.1. *The* $N_{i,k}$ *defined in* (2.1) *satisfy*

(2.2) $0 < N_{i,k}(x) \leqslant 1$ *for* $x \in (x_i, x_{i+k})$ *and* $N_{i,k}(x) = 0$ *otherwise*;

(2.3) $\{N_{i,k}\}_{i=j}^{j+r}$ *is linearly independent over* $[x_{j+k-1}, x_{j+r+1}]$, *for any* $r \geqslant k-1$ *and any* $1-k \leqslant j \leqslant N-r-1$;

(2.4) $\{N_{i,k}\}_{i=1-k}^{N-1}$ *spans* $\mathscr{S}_{k,\pi}$;

(2.5) $\sum_{i=1-k}^{N-1} \xi_i^{(\mu)} N_{i,k}(x) = U_\mu(x) = x^{\mu-1}, \qquad \mu = 1, 2,\ldots, k$, *where*

$$\xi_i^{(\mu)} = (-1)^{\mu-1} \frac{(\mu-1)!}{(k-1)!}\, \psi_i^{(k-\mu)}(0) = \frac{\operatorname{sym}_{\mu-1}(x_{i+1}, \ldots, x_{i+k-1})}{\binom{k-1}{\mu-1}},$$

where $\operatorname{sym}_{\mu-1}(x_{i+1}, \ldots, x_{i+k-1})$ *and* ψ_i *are defined by*

$$\psi_i(x) = (x - x_{i+1}) \cdots (x - x_{i+k-1}) = \sum_{\mu=1}^{k} (-1)^{\mu-1} x^{k-\mu} \operatorname{sym}_{\mu-1}(x_{i+1}, \ldots, x_{i+k-1});$$

(2.6) *Suppose* $x_m \leqslant x < x_{m+1}$ *and* $i \leqslant m < i+k$. *Fix* $0 < r < k$. *If* $x = x_m$, *suppose also that* x_m *is of multiplicity at most* $k-r-1$. *Then* $N_{i,k}^{(r)}(x)$ *exists, and*

$$|N_{i,k}^{(r)}(x)| \leqslant \frac{\Gamma_{kr}}{\Delta_{i,m,k-1} \cdots \Delta_{i,m,k-r}},$$

where for $j = k-r,\ldots, k-1$ *we define* Δ_{imj} *as the minimum of* $x_{\nu+j} - x_\nu$ *over* ν *such that* $x_i \leqslant x_\nu \leqslant x < x_{\nu+j} \leqslant x_{i+k}$, *and where*

$$\Gamma_{kr} = \frac{(k-1)!}{(k-r-1)!} \begin{bmatrix} r \\ [r/2] \end{bmatrix}$$

with $[r/2]$ = *greatest integer less than or equal to* $r/2$.

Proof. For (2.2) and (2.4) see Curry and Schoenberg [7]. De Boor and Fix [6] proved (2.3). Relation (2.5) has been proved by several authors; for a proof with the ξ_i's as given here see de Boor [5].

The estimate (2.6) is a refinement of one in de Boor and Fix [6]. To prove it, we first note that

$$N_{i,k}^{(r)}(x) = \frac{(x_{i+k} - x_i)(k-1)!}{(k-r-1)!} [x_i, \ldots, x_{i+k}]\, G_{k-r}(\cdot; x).$$

Using Lemma 2.2 below with $\omega = k+1$, $\mu = k-r$ we obtain

$$| N_{i,k}^{(r)}(x)| \leqslant \frac{(k-1)!}{(k-r-1)!} \sum_{\nu=0}^{r} \frac{\binom{r}{\nu} |[x_{\nu+i}, \ldots, x_{\nu+i+k-r}]\, G_{k-r}(\cdot; x)|}{\Delta_{imk-1} \cdots \Delta_{imk-r+1}},$$

$$\leqslant \frac{(k-1)! \binom{r}{[r/2]} \sum_{\nu=0}^{r} N_{\nu+i,k-r}(x)}{(k-r-1)!\, \Delta_{imk-1} \cdots \Delta_{imk-r}}.$$

Since $\sum N_{\nu+i,k-r}(x) \leqslant 1$, (2.6) follows. ▮

LEMMA 2.2. *Let* $0 \leqslant \mu \leqslant \omega - 2$. *Then for any* $\xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_\omega$ *with* $\gamma_j = \min_{1 \leqslant \nu \leqslant \omega - j} | \xi_{\nu+j} - \xi_\nu | > 0$, $j = \mu+1, \ldots, \omega-1$, *we have*

$$|[\xi_1, \ldots, \xi_\omega]f| \leqslant \left\{ \sum_{\nu=0}^{\omega-\mu-1} \binom{\omega-\mu-1}{\nu} |[\xi_{\nu+1}, \ldots, \xi_{\nu+\mu+1}]f| \right\} \Big/ (\gamma_{\omega-1} \cdots \gamma_{\mu+1}). \tag{2.7}$$

Proof. Since

$$|[\xi_1, \ldots, \xi_\omega]f| \leqslant \frac{|[\xi_2, \ldots, \xi_\omega]f| + |[\xi_1, \ldots, \xi_{\omega-1}]f|}{| \xi_\omega - \xi_1 |},$$

and $| \xi_\omega - \xi_1 | \geqslant \gamma_{\omega-1}$, we have (2.7) for $\mu = \omega - 2$. Now suppose it holds for $0 \leqslant \mu \leqslant \omega - 2$. Then

$$|[\xi_1, \ldots, \xi_\omega]f|$$
$$\leqslant \frac{\sum_{\nu=0}^{\omega-\mu-1} \binom{\omega-\mu-1}{\nu} \left\{ \frac{|[\xi_{\nu+2}, \ldots, \xi_{\nu+\mu+1}]f| + |[\xi_{\nu+1}, \ldots, \xi_{\nu+\mu}]f|}{\xi_{\nu+\mu+1} - \xi_{\nu+1}} \right\}}{\gamma_{\omega-1} \cdots \gamma_{\mu+1}}.$$

Now $(\xi_{\nu+\mu+1} - \xi_{\nu+1}) \geqslant \gamma_\mu$ and combining the sums yields (2.7) for μ replaced by $\mu - 1$. Thus we have proved (2.7) for all $0 \leqslant \mu \leqslant \omega - 2$ by induction. ▮

3. Operators Which Reproduce Polynomials

Let $\mathscr{F}$ be a linear space of real-valued functions on $[a, b]$, and let $\{\lambda_i\}_{k-1}^{N-1}$ be a set of linear functionals λ_i: $\mathscr{F} \to \mathbb{R}$. Given $f \in \mathscr{F}$, we construct an approximation Qf to f by

$$Qf(x) = \sum_{i=1-k}^{N-1} \lambda_i f N_{i,k}(x). \tag{3.1}$$

Q defines a linear operator mapping $\mathscr{F}$ into $\mathscr{S}_{k,\pi}$. Suppose $\mathscr{F}$ contains the class of polynomials $\mathscr{P}_l$ of degree at most $l - 1$ for some $1 \leqslant l \leqslant k$. In this section we study the choices of $\{\lambda_i\}$ which yield

$$Qg = g \qquad \text{for all } g \in \mathscr{P}_l. \tag{3.2}$$

Lemma 3.1. *An operator Q defined as in* (3.1) *satisfies* (3.2) *if and only if*

$$\lambda_i U_\mu = \xi_i^{(\mu)}, \qquad \mu = 1, 2,..., l;\ i = 1 - k,..., N - 1, \tag{3.3}$$

where $U_\mu(x) = x^{\mu-1}$ and $\xi_i^{(\mu)}$ are as in (2.5).

Proof. Fix $1 \leqslant \mu \leqslant l$. By (2.5), $U_\mu(x) = \sum_{i=1-k}^{N-1} \xi_i^{(\mu)} N_{i,k}(x)$ while $QU_\mu(x) = \sum_{i=1-k}^{N-1} \lambda_i U_\mu N_{i,k}(x)$. Since $\{N_{i,k}\}_{1-k}^{N-1}$ is linearly independent, $U_\mu \equiv QU_\mu$ if and only if $\lambda_i U_\mu = \xi_i^{(\mu)}$, $i = 1 - k,..., N - 1$. ∎

It is convenient to construct λ_i satisfying (3.3) from given linear functionals $\{\lambda_{ij}\}_{j=1}^{l}$. We formalize this in a corollary.

Corollary 3.2. *Suppose that for each $i = 1 - k,..., N - 1$, $\{\lambda_{ij}\}_{j=1}^{l}$ is a set of linear functionals defined on $\mathscr{F}$ such that*

$$\det(\lambda_{ij} U_\mu)_{j,\mu=1}^{l} \neq 0. \tag{3.4}$$

Let $\{\alpha_{ij}\}_{j=1}^{l}$ be the solution of

$$\sum_{j=1}^{l} \alpha_{ij} \lambda_{ij} U_\mu = \xi_i^{(\mu)}, \qquad \mu = 1, 2,..., l. \tag{3.5}$$

Then $\lambda_i = \sum_{j=1}^{l} \alpha_{ij} \lambda_{ij}$ satisfy (3.3), *and the corresponding Q satisfies* (3.2).

For any prescribed $\{\lambda_{ij}\}_{j=1}^{l}$ satisfying (3.4), the system (3.5) can always be uniquely solved for the $\{\alpha_{ij}\}_{j=1}^{l}$. This will be especially easy if λ_{ij} have the property that $\lambda_{ij} U_\mu = 0$ for $\mu = 1, 2,..., j - 1$ and $j = 1, 2,..., l$. Then the system is lower triangular and

$$\alpha_{i1} = 1/\lambda_{i1} U_1 \qquad (\xi_i^{(1)} = 1),$$

$$\alpha_{ij} = \left(\xi_i^{(j)} - \sum_{\nu=1}^{j-1} \alpha_{i\nu} \lambda_{i\nu} U_j\right) \Big/ \lambda_{ij} U_j, \qquad j = 2, 3,..., l. \tag{3.6}$$

We point out that in this case the $\{\alpha_{ij}\}$ do not depend on l; i.e., if we have the $\{\alpha_{ij}\}_{j=1}^{l}$, then to solve (3.5) with $l+1$ we need only compute one new coefficient.

The next theorem contains an explicit expression for the $\{\alpha_{ij}\}$ satisfying (3.5) which will be useful for obtaining error bounds for $f - Qf$. We recall that the symmetric functions $\text{sym}_i(\xi_1, \ldots, \xi_e)$ are defined implicitly by $(x+\xi_1)\cdots(x+\xi_e) = \sum_{\nu=1}^{e+1} \text{sym}_{e-\nu+1}(\xi_1, \ldots, \xi_e)\, x^{\nu-1}$.

THEOREM 3.3. *For $i = 1-k, \ldots, N-1$ let $\{\lambda_{ij}\}_{j=1}^{l}$ satisfy (3.4), and suppose $\{p_{ij}\}_{j=1}^{l}$ are polynomials of degree at most $l-1$ such that*

$$\lambda_{i\mu} p_{ij} = \delta_{j\mu}, \qquad j, \mu = 1, 2, \ldots, l. \tag{3.7}$$

Then if $p_{ij}(x) = \sum_{\nu=1}^{q_{ij}+1} a_{ij\nu} x^{\nu-1}$ with $0 \leqslant q_{ij} \leqslant l-1$, the solution of (3.5) is given by

$$\alpha_{ij} = \sum_{\nu=1}^{q_{ij}+1} a_{ij\nu} \xi_i^{(\nu)}. \tag{3.8}$$

In particular, if $p_{ij}(x) = c_{ij}(x - z_{ij1}) \cdots (x - z_{ijq_{ij}})$, then

$$\alpha_{ij} = c_{ij} \sum_{\nu=0}^{q_{ij}} (-1)^{\nu} \frac{\text{sym}_{\nu}(z_{ij1}, \ldots, z_{ijq_{ij}})\, \text{sym}_{q_{ij}-\nu}(x_{i+1}, \ldots, x_{i+k-1})}{\binom{k-1}{q_{ij}-\nu}}. \tag{3.9}$$

Proof. It may be verified directly that the α_{ij} in (3.8) satisfy (3.5). To see how this choice arose, we multiply the νth equation of (3.5) by $a_{ij\nu}$ and sum over $\nu = 1, 2, \ldots, l$. ▌

We conclude this section with several examples.

EXAMPLE 3.4. Fix $1 \leqslant d \leqslant l \leqslant k$. For each $i = 1-k, \ldots, N-1$ let $a \leqslant \tau_{i1}, \ldots, \tau_{il} \leqslant b$ be such that at most d of the $\{\tau_{i1}, \ldots, \tau_{il}\}$ are equal to any one value. Let $\lambda_{ij} f = [\tau_{i1}, \ldots, \tau_{ij}]f$, $j = 1, 2, \ldots, l$, where if equal τ's are involved; then the divided difference is interpreted in the usual extended sense involving derivatives (cf., e.g., Isaacson and Keller [9, pp. 246 ff.]). It is well known that $\{\lambda_{ij}\}_{j=1}^{l}$ satisfy (3.4) and moreover, (3.7) holds with $p_{ij}(x) = (x - \tau_{i1}) \cdots (x - \tau_{ij-1})$, $j = 1, 2, \ldots, l$. Thus if $\{\alpha_{ij}\}_{j=1}^{l}$ are given by (3.8), then the approximation scheme

$$Qf(x) = \sum_{i=1-k}^{N-1} \sum_{j=1}^{l} \alpha_{ij} [\tau_{i1}, \ldots, \tau_{ij}] f\, N_{i,k}(x) \tag{3.10}$$

satisfies (3.2); i.e., is exact for $\mathscr{P}_l$. We emphasize that with $d = 1$ in this example, the construction of Qf in (3.10) depends only on values of f at the τ_{ij}, and not on any derivatives of f. In general, if $d \geq 1$, Q is defined for any $f \in C^{d-1}[a, b]$.

For convenience we list α_{ij} for $j = 1, 2, 3, 4$. We have

$$\begin{aligned}
\alpha_{i1} &= 1\\
\alpha_{i2} &= \xi_i^{(2)} - \tau_{i1}\\
\alpha_{i3} &= \xi_i^{(3)} - (\tau_{i1} + \tau_{i2})\, \xi_i^{(2)} + \tau_{i1}\tau_{i2}\\
&= \xi_i^{(3)} - (\tau_{i1} + \tau_{i2})\, \alpha_{i2} - \tau_{i1}^2 .\\
\alpha_{i4} &= \xi_i^{(4)} - (\tau_{i1} + \tau_{i2} + \tau_{i3})\, \xi_i^{(3)} + (\tau_{i1}\tau_{i2} + \tau_{i1}\tau_{i3} + \tau_{i2}\tau_{i3})\, \xi_i^{(2)} - \tau_{i1}\tau_{i2}\tau_{i3}\\
&= \xi_i^{(4)} - (\tau_{i1} + \tau_{i2} + \tau_{i3})\, \alpha_{i3} - (\tau_{i1}^2 + \tau_{i1}\tau_{i2} + \tau_{i2}^2)\, \alpha_{i2} - \tau_{i1}^3 .
\end{aligned}$$

We note that if τ_{i1} is chosen to be $\xi_i^{(2)} = (x_{i+1} + \cdots + x_{i+k-1})/(k-1)$, then $\alpha_{i2} = 0$. Thus for example, with $l = 2$,

$$Qf(x) = \sum_{i=1-k}^{N-1} f(\xi_i^{(2)})\, N_{i,k}(x). \tag{3.11}$$

This is precisely the variation-diminishing spline approximation method o Marsden and Schoenberg [12] which reproduces $\mathscr{P}_2$.

If we select $\tau_{i1} = \xi_i^{(2)}$, then α_{i3} and α_{i4} also simplify to

$$\begin{aligned}
\alpha_{i3} &= \xi_i^{(3)} - (\xi_i^{(2)})^2,\\
\alpha_{i4} &= \xi_i^{(4)} - \xi_i^{(2)}\xi_i^{(3)} - (\tau_{i2} + \tau_{i3})\alpha_{i3} .
\end{aligned}$$

EXAMPLE 3.5. Let $l = k = d$ in Example 3.4. Then we write $\tau_i = \tau_{i1} = \cdots = \tau_{il}$ and note that $p_{ij}(x) = (x - \tau_i)^{j-1}$, and by (3.8) and (2.5),

$$\alpha_{ij} = \sum_{\mu=1}^{j} \frac{(-1)^{\mu-1}\psi_i^{(k-\mu)}(0)(j-1)!}{(k-1)!\,(j-\mu)!}\,(-\tau_i)^{j-\mu} = \frac{(-1)^{j-1}\psi_i^{(k-j)}(\tau_i)(j-1)!}{(k-1)!},$$

$j = 1, 2,..., k$. Then (3.8) becomes

$$Qf(x) = \sum_{i=1-k}^{N-1} \sum_{j=1}^{k} \alpha_{ij} f^{(j-1)}(\tau_i)\, N_{i,k}(x), \tag{3.12}$$

with $\{\alpha_{ij}\}$ given above. This is precisely the quasi-interpolant discussed in de Boor and Fix [6]. It is defined on $C^{k-1}[a, b]$ and is exact for $\mathscr{P}_k$, and even for $\mathscr{S}_{k,\pi}$.

EXAMPLE 3.6. Fix $1 \leqslant l \leqslant k$. For each $i = 1-k,..., N-1$ let $\hat{w}_i(x) \in C(-1, 1)$ be given, where $\hat{w}_i(x) > 0$ for $x \in (-1, 1)$. Suppose $\hat{p}_{ij} \in \mathscr{P}_j$ are the associated orthogonal polynomials; i.e.,

$$\int_{-1}^{1} \hat{w}_i(y)\, \hat{p}_{i\nu}(y)\, \hat{p}_{i\mu}(y)\, dy = \delta_{\nu\mu} h_{i\nu}, \qquad \nu, \mu = 1, 2,..., l. \tag{3.13}$$

The polynomial $\hat{p}_{ij}$ has $j-1$ distinct zeros in $(-1, 1)$; say $\hat{p}_{ij}(y) = k_{ij}(y - \xi_{ij1}) \cdots (y - \xi_{ijj-1})$ with $k_{ij} \neq 0$ and $\xi_{ij1},..., \xi_{ijj-1} \in (-1, 1)$. Typically, we would choose $\hat{w}_i$ to yield classical orthogonal polynomials.

Now given any $\alpha_i < \beta_i$ and $f \in L_1[\alpha_i, \beta_i]$, we can define

$$\lambda_{ij} f = \int_{-1}^{1} \hat{w}_i(y)\, \hat{p}_{ij}(y) f\left(\frac{\beta_i - \alpha_i}{2}\, y + \frac{\alpha_i + \beta_i}{2}\right) dy, \quad j = 1, 2,..., l. \tag{3.14}$$

If

$$p_{ij}(x) = \hat{p}_{ij}\left(\frac{2x - \alpha_i - \beta_i}{\beta_i - \alpha_i}\right)\Big/ h_{ij}, \qquad j = 1, 2,..., l, \tag{3.15}$$

then (3.7) is satisfied. Thus with α_{ij} given by (3.8) (where $q_{ij} = j - 1$), the B-spline approximation method

$$Qf(x) = \sum_{i=1-k}^{N-1} \sum_{j=1}^{l} \alpha_{ij} \left(\int_{-1}^{1} \hat{w}_i(y)\, \hat{p}_{ij}(y) f\left(\frac{\beta_i - \alpha_i}{2}\, y + \frac{\alpha_i + \beta_i}{2}\right) dy\right) N_{i,k}(x) \tag{3.16}$$

is defined for $f \in L_1[a, b]$ and is exact for $\mathscr{P}_l$.

For later convenience, we note that

$$p_{ij}(x) = c_{ij}(x - z_{ij1}) \cdots (x - z_{ijj-1}) \tag{3.17}$$

with

$$c_{ij} = \frac{k_{ij}}{h_{ij}} \left(\frac{2}{\beta_i - \alpha_i}\right)^{j-1} \qquad \text{and} \qquad z_{ij1},..., z_{ijj-1} \in (\alpha_i, \beta_i).$$

We also note that taking

$$\lambda_{ij} f = \int_{-1}^{1} y^{j-1}\, \hat{w}_i(y) f\left(\frac{\beta_i - \alpha_i}{2}\, y + \frac{\alpha_i + \beta_i}{2}\right) dy, \qquad j = 1, 2,..., l$$

(the weighted moments of f over $(-1, 1)$) is equivalent to the choice (3.14); that is, they lead to the same Q.

4. Some Lemmas

In this section we provide some tools which will be useful for estimating how well B-spline approximation methods approximate smooth functions. In particular, we shall be interested in the quantities

$$E_{r,s}(t) = \begin{cases} D^r(f - Qf)(t), & 0 \leqslant r < s \\ D^r Qf(t), & s \leqslant r < k, \end{cases} \tag{4.1}$$

where D^r is the rth derivative operator and s is an integer with $1 \leqslant s \leqslant k$. We have introduced the parameter s since often the B-spline approximation Qf will have more derivatives than f. In fact, if Qf is given by (3.1) and if m is an integer such that $x_m \leqslant t < x_{m+1}$, then

$$Qf(t) = \sum_{i=m+1-k}^{m} \lambda_i f N_{ik}(t) \tag{4.2}$$

and

$$D^r Qf(t) = \sum_{i=m+1-k}^{m} \lambda_i f D^r N_{ik}(t). \tag{4.3}$$

We recall (cf. (2.6)) that $D^r N_{ik}(t)$ exists for all $t \notin \pi$, and also if $t = x_m \in \pi$, provided x_m has multiplicity at most $k - r - 1$.

Our first result is a basic comparison lemma which is designed to exploit the property that Q reproduces polynomials. The idea has been much used; cf., e.g., de Boor and Fix [6].

Lemma 4.1. *Suppose Q is defined on a class of functions containing $\mathscr{P}_l$, and suppose* (3.2) *holds* (*i.e., Q reproduces $\mathscr{P}_l$*). *Then for any polynomial $g \in \mathscr{P}_s$ and any f such that $D^r f(t)$ exists,* $0 \leqslant r < s \leqslant l \leqslant k$,

$$E_{r,s}(t) = \begin{cases} D^r R(t) - D^r QR(t), & 0 \leqslant r < s \\ D^r QR(t), & s \leqslant r < k, \end{cases} \tag{4.4}$$

where $R(x) = f(x) - g(x)$.

Proof. For $0 \leqslant r < s$,

$$D^r(f - Qf) = D^r(f - g) + D^r(Qg - Qf) = D^r R - D^r QR,$$

since $Qg = g$. For $s \leqslant r < k$,

$$D^r Qf = D^r Qf - D^r g = D^r Qf - D^r Qg = D^r QR,$$

since $D^r g = 0$. ∎

Lemma 4.1 reduces the problem of estimating $|E_{r,s}(t)|$ to obtaining estimates for $|D^r R(t)|$ and $|D^r QR(t)|$. The first of these is usually easy (e.g., if g is the Taylor expansion at t, then R and its derivatives are 0 at t). For the second term we have, by (4.3),

$$|D^r QR(t)| \leqslant \sum_{i=m+1-k}^{m} |\lambda_i R|\, |D^r N_{ik}(t)|.$$

We have bounds on $|D^r N_{ik}(t)|$ in (2.6); it remains to study $|\lambda_i R|$. If $\lambda_i R = \sum_{j=1}^{l} \alpha_{ij}\lambda_{ij} R$ with $\{\alpha_{ij}\}$ satisfying (3.5), we have

$$|\lambda_i R| \leqslant \sum_{j=1}^{l} |\alpha_{ij}|\, |\lambda_{ij} R|.$$

We will have to estimate $|\lambda_{ij} R|$ for specific choices of λ_{ij} and R (see Sections 5 and 6). In the remainder of this section we concentrate on the α_{ij}.

LEMMA 4.2. *Let $\xi = (\xi_1, ..., \xi_e)$ and $\eta = (\eta_1, ..., \eta_d)$ be vectors of real numbers with $e \geqslant d$. Define*

$$\phi(\xi; \eta) = (e-d)! \sum (\eta_1 + \xi_{i_1}) \cdots (\eta_d + \xi_{i_d}), \qquad (4.5)$$

where the sum is taken over all choices of distinct $\xi_{i_1}, ..., \xi_{i_d}$ from $\xi_1, ..., \xi_e$. (This is a sum of $e!/(e-d)!$ terms). Then

$$\phi(\xi, \eta) = \sum_{\mu=0}^{d} (e-d+\mu)!\,(d-\mu)!\, \mathrm{sym}_{d-\mu}(\xi_1, ..., \xi_e)\, \mathrm{sym}_\mu(\eta_1, ..., \eta_d). \qquad (4.6)$$

Proof. It may be verified that for $\nu = 0, 1, ..., d$,

$$\frac{\partial\phi}{\partial\eta_{j_1} \cdots \partial\eta_{j_\nu}}(\xi; 0) = (e-d+\nu)!\,(d-\nu)!\, \mathrm{sym}_{d-\nu}(\xi_1, ..., \xi_e),$$

for any choice of distinct $\eta_{j_1}, ..., \eta_{j_\nu}$ from $\eta_1, ..., \eta_d$ while

$$\partial\phi(\xi; 0)/(\partial\eta_{j_1} \cdots \partial\eta_{j_\nu}) = 0$$

if any two of the $\eta_{j_1}, ..., \eta_{j_\nu}$'s are equal. Now (4.6) is just the Taylor expansion of $\phi(\xi; \eta)$ with respect to the η-variable about $\eta = 0$. ∎

LEMMA 4.3. *Suppose $\{\lambda_{ij}\}_{j=1}^{l}$ and $\{p_{ij}\}_{j=1}^{l}$ are as in Theorem 3.3. Then*

$$\alpha_{ij} = c_{ij} \frac{(k - q_{ij} - 1)!}{(k-1)!} \sum (x_{\nu_1} - z_{ij1}) \cdots (x_{\nu_{q_{ij}}} - z_{ijq_{ij}}), \qquad (4.7)$$

where the sum is taken over all choices of distinct $\nu_1,\ldots,\nu_{q_{ij}}$ *from* $i+1,\ldots, i+k-1$. *(This is a sum of* $(k-1)!/(k-q_{ij}-1)!$ *terms.)*

Proof. We combine (3.9) with (4.5) and (4.6) with $e = k-1$, $d = q_{ij}$, $\{\xi_j = x_{i+j}\}_{j=1}^{k-1}$, and $\{\eta_\mu = -z_{ij\mu}\}_{\mu=1}^{q_{ij}}$. ∎

LEMMA 4.4. *Suppose* Q *reproduces polynomials* $\mathscr{P}_l$ *and that* $\{\lambda_{ij}\}_{j=1}^{l}$ *and* $p_{ij}(x) = c_{ij}(x - z_{ij1}) \cdots (x - z_{ijq_{ij}})$ *are as in Lemma* 4.3. *Given* t, *let* m *be such that* $x_m \leqslant t < x_{m+1}$. *Then with* R *as in Lemma* 4.1,

$$| D^r QR(t)| \leqslant k\Gamma_{kr} \max_{m+1-k \leqslant i \leqslant m} \sum_{j=1}^{l} | \lambda_{ij} R | \, | c_{ij} | \, A_{ij} , \tag{4.8}$$

where

$$A_{ij} = \max_{\substack{i+1 \leqslant \nu_1,\ldots,\nu_{q_{ij}} \leqslant i+k-1 \\ \nu_1,\ldots,\nu_{q_{ij}} \text{ distinct}}} \frac{| x_{\nu_1} - z_{ij1} | \cdots | x_{\nu_{q_{ij}}} - z_{ijq_{ij}} |}{\Delta_{i,m,k-1} \cdots \Delta_{i,m,k-r}} \tag{4.9}$$

and where Γ_{kr} *and* $\Delta_{im\nu}$ *are defined in* (2.6).

5. Error Bounds for a Method Based on Point Evaluators

Fix integers $1 \leqslant d \leqslant l \leqslant k$. For $i = 1-k,\ldots, N-1$ let $\{\tau_{ij}\}_{j=1}^{l}$ be prescribed real numbers in $[a, b] \cap [x_i, x_{i+k}]$ such that for fixed i at most d of the $\{\tau_{ij}\}_1^l$ are equal to any one value. Throughout this section we will be concerned with the B-spline approximation method

$$Q^E f(x) = \sum_{i=1-k}^{N-1} \sum_{j=1}^{l} \alpha_{ij} \lambda_{ij}^E f \, N_{i,k}(x), \tag{5.1}$$

where

$$\lambda_{ij}^E = [\tau_{i1},\ldots, \tau_{ij}] f \tag{5.2}$$

and α_{ij} are given by (3.8) with $p_{ij}(x) = (x - \tau_{i1}) \cdots (x - \tau_{ij-1})$, $j = 1, 2,\ldots, l$; $i = 1-k,\ldots, N-1$. (This is just Example 3.4 with $\{\tau_{ij}\}_1^l$ restricted to lie in the support of $N_{i,k}$.) This makes Q^E a local approximation scheme; the value of $Q^E f(t)$ depends only on the values of f in a (small) neighborhood of t). We recall Q^E reproduces $\mathscr{P}_l$. It is defined for all $f \in C^{d-1}[a, b]$.

The purpose of this section is to obtain estimates for $| E_{r,s}^E(t)|$, which we define as in (4.1) with $Q = Q^E$. We shall use Lemmas 4.1 and 4.4, so we need to introduce an appropriate R.

For fixed $a \leqslant t \leqslant b$, let m be such that $x_m \leqslant t < x_{m+1}$. We write $I_{i\nu}$ for the smallest closed interval containing $\{\tau_{ij}\}_{j=1}^{\nu}$, and I_m for the smallest closed interval containing $[x_m, x_{m+1}]$ and $\bigcup_{i=m+1-k}^{m} I_{il}$. (The set $\bigcup I_{il}$ contains the support of the $\{\lambda_i\}_{m+1-k}^{m}$ involved in constructing $Qf(t)$). Now for $f \in C^{s-1}(I_m)$ we define

$$R(x) = f(x) - \sum_{i=0}^{s-1} \frac{f^{(i)}(t)}{i!} (x-t)^i. \tag{5.3}$$

By the Taylor series for R we have

$$D^{j-1}R(x) = \frac{D^{s-1}R(\zeta)(x-t)^{s-j}}{(s-j)!}, \qquad j = 1, 2, \ldots, s, \tag{5.4}$$

for some $\zeta(x)$ between x and t. We also note that

$$D^{s-1}R(x) = D^{s-1}f(x) - D^{s-1}f(t). \tag{5.5}$$

Our first task is to estimate $|\lambda_{ij}^{E} R|$. As we will be using Lemma 2.2 we need to introduce parameters describing the spacing of the τ_{ij}. For each integer $1 \leqslant \nu \leqslant l-1$, let

$$\sigma_{ij\nu} = \min_{1 \leqslant \mu \leqslant j-\nu} (\tau_{i\mu+\nu}^{(j)} - \tau_{i\mu}^{(j)}), \tag{5.6}$$

where $\{\tau_{i1}^{(j)}, \ldots, \tau_{ij}^{(j)}\}$ is the nondecreasing rearrangement of $\{\tau_{i1}, \ldots, \tau_{ij}\}$. Since at most d of the τ_{ij} are equal to any one value, $\sigma_{ij\nu} > 0$ for $\nu = d, d+1, \ldots, l-1$. We set

$$\sigma_{is} = \min_{1 \leqslant j \leqslant l} \sigma_{ijs}. \tag{5.7}$$

We will also need parameters describing the spacing of the partition π. Let

$$\bar{\Delta}_m = \max_{m+1-k \leqslant i \leqslant m+k-1} (x_{i+1} - x_i), \tag{5.8}$$

$$\bar{\Delta} = \max_{1-k \leqslant i \leqslant N+k-2} (x_{i+1} - x_i), \tag{5.9}$$

$$\Delta_{m,k-r} = \min_{m+1-k+r \leqslant \nu \leqslant m} (x_{\nu+k-r} - x_\nu), \tag{5.10}$$

and

$$\Delta_{k-r} = \min_{\substack{0 \leqslant m < N-1 \\ x_m < x_{m+1}}} \Delta_{m,k-r}. \tag{5.11}$$

Finally, we define the modulus of continuity of a function $g \in C(I)$ by

$$\omega(g; \Delta; I) = \max_{\substack{x, x+h \in I \\ 0 \leqslant h \leqslant \Delta}} |f(x+h) - f(x)|.$$

LEMMA 5.1. *Let* $1 \leqslant d \leqslant s \leqslant l$, *where d is the maximum multiplicity of the* $\{\tau_{i\nu}\}_{\nu=1}^{j}$ *defining* λ_{ij}^{E}. *Let* $m+1-k \leqslant i \leqslant m$. *Then if* $f \in C^{s-1}(I_m)$,

$$|\lambda_{ij}^{E}R| \leqslant k\omega(D^{s-1}f; \bar{\Delta}_m; I_m)\begin{cases} \dfrac{|\zeta_{ij}-t|^{s-j}}{(j-1)!\,(s-j)!}, & j=1,2,\ldots,s, \\ \dfrac{2^{j-s}}{(s-1)!\,\sigma_{ij,j-1}\cdots\sigma_{ijs}}, & j=s+1,\ldots,l, \end{cases} \tag{5.12}$$

where $\zeta_{ij} \in I_{ij}$.

Proof. For any $x \in I_m$ we have

$$|D^{s-1}f(x) - D^{s-1}f(t)| \leqslant k\omega(D^{s-1}f; \bar{\Delta}_m; I_m). \tag{5.13}$$

Now to prove the first inequality we note that, for $j = 1, 2,\ldots, s$, $\lambda_{ij}^{E}R = D^{j-1}R(\zeta_{ij})/(j-1)!$, where $\zeta_{ij} \in I_{ij} \subset I_m$. Now (5.4), (5.5), and (5.13) yield the result. For the second inequality we use Lemma 2.2 with $\omega = j$ and $\mu = s-1$. Since $\sum_{\nu=0}^{j-s}\binom{j-s}{\nu} = 2^{j-s}$ we obtain

$$|\lambda_{ij}^{E}R| \leqslant 2^{j-s} \max_{0\leqslant\nu\leqslant j-s} \frac{|[\tau_{i,\nu+1},\ldots,\tau_{i,\nu+s}]R|}{\sigma_{i,j,j-1}\cdots\sigma_{i,j,s}}.$$

But $|[\tau_{i\nu+1},\ldots,\tau_{i\nu+s}]R| = |D^{s-1}R(\zeta_{ij\nu})|/(s-1)!$, where $\zeta_{ij\nu} \in I_m$, $\nu = 0, 1,\ldots, j-s$. Thus (5.5) and (5.13) yield the result. ∎

We are now ready for our first error estimates. We begin with local error estimates. Recall that I_m is the smallest interval containing $[x_m, x_{m+1}]$ and the support of $\{\lambda_{ij}\}_{j=1}^{l}$ for $i = m+1-k,\ldots, m$. We write $L_p{}^s[I] = \{f: f^{(s-1)}$ is absolutely continuous on I and $f^{(s)} \in L_p[I]\}$.

THEOREM 5.2. *Let* $1 \leqslant d \leqslant s \leqslant l \leqslant k$ *and* $1 \leqslant q \leqslant \infty$. *If* $f \in C^{s-1}[I_m]$, *then for* $0 \leqslant r < k$,

$$\|E_{rs}^{E}\|_{L_q[x_m,x_{m+1}]} \leqslant K_m \bar{\Delta}_m^{s-r-1+(1/q)}\,\omega(D^{s-1}f; \bar{\Delta}_m; I_m). \tag{5.14}$$

If, moreover, $f \in L_p{}^s[I_m]$, $1 \leqslant p \leqslant \infty$, *then for* $0 \leqslant r < k$,

$$\|E_{rs}^{E}\|_{L_q[x_m,x_{m+1}]} \leqslant K_m \bar{\Delta}_m^{s-r+(1/q)-(1/p)}\,\|D^s f\|_{L_p[I_m]}. \tag{5.15}$$

Here

$$K_m = \frac{k^{s+1}\Gamma_{kr}}{(s-1)!}\left(\frac{\bar{\Delta}_m}{\Delta_{m,k-r}}\right)^r\left[2^{s-1} + \sum_{j=s+1}^{l}(2\rho_m)^{j-s}\right],$$

where

$$\rho_m = \max_{m+1-k\leqslant i\leqslant m} \frac{(x_{i+k}-x_i)}{\sigma_{is}},$$

and $\Gamma_{kr} = (k-1)!/(k-r-1)!\binom{r}{[r/2]}$ *is the constant in* (2.6).

Proof. We use Lemmas 4.1, 4.4, and 5.1. In Lemma 4.4 we take $q_{ij} = j - 1$ and $z_{ij\nu} = \tau_{i\nu}$, $\nu = 1, 2,..., j - 1$. Then for $x_m \leqslant t < x_{m+1}$,

$$| E_{rs}^E(t)| \leqslant k^2 \Gamma_{kr}\, \omega(D^{s-1}f; \bar{\Delta}_m ; I_m) \cdot \max_{m+1-k \leqslant i \leqslant m} \left[\sum_{j=1}^{s} \frac{| \zeta_{ij} - t |^{s-j}}{(j-1)!\,(s-j)!} A_{ij} + \sum_{j=s+1}^{l} \frac{2^{j-s} A_{ij}}{(s-1)!\, \sigma_{ijj-1} \cdots \sigma_{ijs}} \right], \tag{5.16}$$

where

$$A_{ij} = \max_{\substack{i+1 \leqslant \nu_1, \dots, \nu_{j-1} \leqslant i+k-1 \\ \nu_\mu \text{ distinct}}} \frac{| x_{\nu_1} - \tau_{i1} | \cdots | x_{\nu_{j-1}} - \tau_{ij-1} |}{\Delta_{i,m,k-1} \cdots \Delta_{i,m,k-r}} .$$

Now since the $\tau_{i\nu}$'s lie in $[x_i, x_{i+k}]$, $| x_{\nu_\mu} - \tau_{i\mu} | \leqslant | x_{i+k} - x_i | \leqslant \rho_m \sigma_{is}$ and $| x_{\nu_\mu} - \tau_{i\mu} | \leqslant (k - 1) \bar{\Delta}_m$. Since $x_m \leqslant t < x_{m+1}$ and $x_{m+1-k} \leqslant \zeta_{ij} \leqslant x_{m+k}$, we also have $| \zeta_{ij} - t | \leqslant k\bar{\Delta}_m$. Thus (5.16) implies (5.14) for $q = \infty$. Integrating the inequality over $[x_m, x_{m+1})$ proves (5.14) for $1 \leqslant q < \infty$.

Now if $f \in L_p{}^s[I_m]$, then for any x, $x + \theta \in I_m$,

$$| D^{s-1}f(x + \theta) - D^{s-1}f(x)| \leqslant \int_x^{x+\theta} | D^s f(u)|\, du \leqslant \left(\int_x^{x+\theta} | D^s f(u)|^p\, du \right)^{(1/p)} \theta^{1-(1/p)}.$$

Taking the supremum over all $| \theta | \leqslant \bar{\Delta}_m$ yields

$$\omega(D^{s-1}f; \bar{\Delta}_m ; I_m) \leqslant \bar{\Delta}_m^{1-(1/p)} \| D^s f \|_{L_p[I_m]} . \quad ∎$$

These local error estimates lead immediately to the following global result.

THEOREM 5.3. *Let* $1 \leqslant d \leqslant s \leqslant l \leqslant k$ *and* $1 \leqslant q \leqslant \infty$. *If* $f \in C^{s-1}[a, b]$, *then for* $0 \leqslant r < k$,

$$\| E_{rs}^E \|_{L_q[a,b]} \leqslant K \bar{\Delta}^{s-r-1}\, \omega(D^{s-1}f; \bar{\Delta}; [a, b]). \tag{5.17}$$

If $f \in L_p{}^s[a, b]$, $1 \leqslant p \leqslant q$, *then for* $0 \leqslant r < k$,

$$\| E_{rs}^E \|_{L_q[a,b]} \leqslant (2k - 1)\, K \bar{\Delta}^{s-r+(1/q)-(1/p)} \| D^s f \|_{L_p[a,b]} . \tag{5.18}$$

Here

$$K = \frac{k^{s+1} \Gamma_{kr}}{(s-1)!} \left(\frac{\bar{\Delta}}{\Delta_{k-r}} \right)^r \left[2^{s-1} + \sum_{j=s+1}^{l} (2\rho)^{j-s} \right],$$

where

$$\rho = \max_{1-k \leqslant i \leqslant N-1} \frac{(x_{i+k} - x_i)}{\sigma_{is}}.$$

Proof. The assertion (5.17) follows immediately from (5.14). Indeed, $\bar{\Delta}_m \leqslant \bar{\Delta}$ for all m. Also, if m is such that $x_m < x_{m+1}$, then $\Delta_{m,k-r} > 0$ and the quantity Δ_{k-r} defined in (5.11) is also positive with $\Delta_{m,k-r} \geqslant \Delta_{k-r}$. Finally, since $\rho_m \leqslant \rho$, $K_m \leqslant K$.

Now by Theorem 5.2, if $f \in L_p^s[a, b]$,

$$\| E_{rs}^E \|_{L_q[x_m, x_{m+1}]} \leqslant K\bar{\Delta}^{s-r+(1/q)-(1/p)} \| D^s f \|_{L_p[I_m]}.$$

Raising this to the qth power and summing over the ν such that $x_{m_\nu} < x_{m_\nu+1}$ yields

$$\left(\sum_\nu \| E_{rs}^E \|_{L_q[x_{m_\nu}, x_{m_\nu+1}]}^q\right)^{(1/q)} \leqslant K\bar{\Delta}^{s-r+(1/q)-(1/p)} \left(\sum_\nu \| D^s f \|_{L_p[I_{m_\nu}]}^q\right)^{(1/q)}.$$

But for $p \leqslant q$, Jensen's inequality (see, e.g., [16]) yields

$$\left(\sum_\nu \| D^s f \|_{L_p[I_{m_\nu}]}^q\right)^{(1/q)} \leqslant \left(\sum_\nu \| D^s f \|_{L_p[I_{m_\nu}]}^p\right)^{(1/p)} \leqslant (2k-1) \| D^s f \|_{L_p[a,b]},$$

since $I_{m_\nu} \subset [x_{m_\nu+1-k}, x_{m_\nu+k}]$ so any piece of $[a, b]$ is added into the sum at most $(2k-1)$ times. ∎

The error bounds in (5.18) may be compared with the classical bounds for spline interpolation (see, e.g., [16]). In particular, the orders are best possible. The constants, however, are not best possible. We have exhibited them primarily to show clearly on what they depend.

It is of interest to examine the question of when the constants K_m and K in the above theorems are independent of the mesh ratios $\bar{\Delta}_m/\Delta_{m,k-r}$ or $\bar{\Delta}/\Delta_{k-r}$ and of the constants ρ_m or ρ. This question depends on the placement of the supports of the λ_{ij} within the support $[x_i, x_{i+k}]$ of the B-spline $N_{i,k}$.

For $1 - k \leqslant i \leqslant N - 1$ and $0 \leqslant r < k$, we define

$$J_{ir} = \bigcap_{\substack{i \leqslant \nu \leqslant i+r \\ |x_{\nu+k-r} - x_\nu| > 1 \\ x_{\nu+k-r} \geqslant x_1 \\ x_\nu \leqslant x_{N-1}}} [x_\nu, x_{\nu+k-r}]. \tag{5.19}$$

(Note that x_1 and x_{N-1} are (as in Section 2) the first point in the partition π bigger than a and the last point smaller than b, respectively). We note that with simple knots,

$$J_{ir} = \begin{cases} [x_{i+r}, x_1], & i = 1-k,..., r-k \\ [x_{i+r}, x_{i+k-r}], & i = r-k+1,..., N-r-1 \\ [x_{N-1}, x_{i+k-r}], & i = N-r,..., N-1. \end{cases} \tag{5.20}$$

If x_i or x_{i+k} is a multiple knot, these intervals are even longer. Thus a sufficient condition for J_{ir} to be a nontrivial interval is that $2r < k$.

The following lemmas will be used to estimate (5.16). We recall that I_{ij} denotes the smallest interval containing the $\{\tau_{i1},..., \tau_{ij}\}$.

LEMMA 5.4. *Fix* $0 \leqslant \mu \leqslant r \leqslant s-1$. *Suppose*

$$I_{il} \subset J_{ir} \cap [a, b], \qquad i = m+1-k,..., m. \tag{5.21}$$

Then for $i = m+1-k,..., m$

$$\frac{|x_{\epsilon_1} - \tau_{i\theta_1}| \cdots |x_{\epsilon_\mu} - \tau_{i\theta_\mu}|}{\Delta_{i,m,k-1}\Delta_{i,m,k-\mu}} \leqslant 1, \tag{5.22}$$

for all choices of distinct $\epsilon_1,..., \epsilon_\mu \in \{i+1,..., i+k-1\}$ *and* $\theta_1,..., \theta_\mu \in \{1,..., l\}$.

Proof. Let $i \leqslant \gamma \leqslant i+\mu$ be such that $[x_m, x_{m+1}] \subset [x_\gamma, x_{\gamma+k-\mu}]$ and $x_{\gamma+k-\mu} - x_\gamma = \Delta_{i,m,k-\mu}$. Now at most $\mu - 1$ of the $x_{\epsilon_1},..., x_{\epsilon_\mu}$ lie outside of $[x_\gamma, x_{\gamma+k-\mu}]$, so at least one is inside. Moreover, all of the $\tau_{i\theta}$'s are in $I_{il} \subset J_{ir}$, and since by definition $J_{ir} \subset [x_\gamma, x_{\gamma+k-\mu}]$, one of the factors $|x_{\epsilon_\nu} - \tau_{i\theta_\nu}|$ is less than or equal to $\Delta_{i,m,k-\mu}$. But then (5.22) follows by induction. ∎

The next lemma is useful for the terms in the first sum in (5.16).

LEMMA 5.5. *Fix* $0 \leqslant r \leqslant s-1$, *and suppose* $x_m \leqslant t < x_{m+1}$. *Suppose* (5.21) *holds. Then for* $i = m+1-k,..., m$ *and* $j = 1, 2,..., s$,

$$\frac{|\zeta_{ij} - t|^{s-j} |x_{\nu_1} - \tau_{i1}| \cdots |x_{\nu_{j-1}} - \tau_{ij-1}|}{\Delta_{i,m,k-1} \cdots \Delta_{i,m,k-r}} \leqslant (k\bar{\Delta}_m)^{s-r-1}, \tag{5.23}$$

for all choices of $\zeta_{ij} \in I_{ij}$ *and all choices of distinct* $\nu_1,..., \nu_{j-1}$ *from* $\{i+1,..., i+k-1\}$.

Proof. Let γ be such that $x_i \leqslant x_\gamma \leqslant t < x_{\gamma+k-r} \leqslant x_{i+k}$ and $\Delta_{i,m,k-r} =$

$(x_{\gamma+k-r} - x_\gamma)$. Then by (5.21), $I_{ij} \subset J_{ir} \subset [x_\gamma, x_{\gamma+k-r}]$, so $|\zeta_{ij} - t| \leqslant \Delta_{i,m,k-r} \leqslant \cdots \leqslant \Delta_{i,m,k-r+s-j-1}$. Now we apply Lemma 5.4 to yield

$$\frac{|x_{\nu_1} - \tau_{i1}| \cdots |x_{\nu_{r-s+j}} - \tau_{ir-s+j}|}{\Delta_{i,m,k-1} \cdots \Delta_{i,m,k-r+s-j}} \leqslant 1.$$

The remaining $s - r - 1$ factors are each bounded by $k\bar{\Delta}_m$. ▮

For the terms in the second sum in (5.16) we have

LEMMA 5.6. *Fix* $0 \leqslant r \leqslant s - 1 \leqslant l - 2$ *and* $x_m \leqslant t < x_{m+1}$. *Suppose* $2r \leqslant s + 1$. *Assume, further, that* (5.21) *holds and* $\sigma_{is} > 0$, $i = m + 1 - k, \ldots, m$. *Then for* $i = m + 1 - k, \ldots, m$ *and* $j = s + 1, \ldots, l$,

$$\frac{|x_{\nu_1} - \tau_{i1}| \cdots |x_{\nu_{j-1}} - \tau_{ij-1}|}{\sigma_{ijj-1} \cdots \sigma_{ijs}\Delta_{i,m,k-1} \cdots \Delta_{i,m,k-r}} \leqslant (\rho_m{}^*)^{j-s} (k\bar{\Delta}_m)^{s-r-1}, \qquad (5.24)$$

for all choices of distinct $\nu_1, \ldots, \nu_{j-1}$ *from* $\{i + 1, \ldots, i + k - 1\}$, *where*

$$\rho_m{}^* = \max_{m+1-k \leqslant i \leqslant m} \frac{|J_{ir}|}{\sigma_{is}}, \qquad (5.25)$$

and $|J_{ir}|$ = *length of* J_{ir}.

Proof. Fix $s + 1 \leqslant j \leqslant l$. If $r = 0$, all of the x_ν's in (5.24) are in J_{ir}. If $r > 0$, the fact that J_{ir} contains $[x_{i+r}, x_{i+k-r}]$ implies at most $2r - 2$ of the $\{x_{i+1}, \ldots, x_{i+k-1}\}$ lie outside of J_{ir}, so at least $j - 1 - 2r + 2 \geqslant j - s$ of the $\{x_{\nu_1}, \ldots, x_{\nu_{j-1}}\}$ lie in J_{ir}. Since by (5.21) the τ's are also in J_{ir}, (5.25) implies at least $j - s$ of the $|x - \tau|$ factors in (5.24) are bounded by $\rho_m{}^*\sigma_{is}$. We are left with $s - 1 \geqslant r$ factors in the numerator, and we may apply Lemma 5.4. ▮

THEOREM 5.7. *Suppose in Theorem* 5.2 *that* (5.21) *holds. In addition, suppose* $r \leqslant s - 1$ *and that* $2r \leqslant s + 1$ *if* $s < l$. *Then* (5.14) *and* (5.15) *hold, with* K_m *replaced by*

$$K_m{}^* = \frac{k^{s-r+1}\Gamma_{kr}}{(s-1)!}\left[2^{s-1} + \sum_{j=s+1}^{l} (2\rho_m{}^*)^{j-s}\right].$$

Proof. We simply apply Lemmas 5.5 and 5.6 to (5.16). ▮

We emphasize that if $s = l$ in the above, then $K_m{}^*$ depends only on k, r, and s. For $s < l$, it would be reasonable to choose the $\tau_{i1}, \ldots, \tau_{il}$ equally spaced throughout $J_{ir} \cap [a, b]$ with τ_{i1} = left endpoint and τ_{il} = right endpoint. Then if the $\{x_{1-k}, \ldots, x_{-1}\}$ and $\{x_{N+1}, \ldots, x_{N+k-1}\}$ in the extended partition have been chosen such that $x_{j+1} - x_j \leqslant x_1 - x_0$, $j = 1 - k,..., -1$, and $x_{j+1} - x_j \leqslant x_N - x_{N-1}$, $j = N,..., N + k - 2$, the constant $\rho_m{}^*$ in (5.25) satisfies $\rho_m{}^* \leqslant (k - r)(l - 1)/s$.

THEOREM 5.8. *Suppose in Theorem* 5.3 *that*

$$I_{il} \subset J_{ir} \cap [a, b], \qquad i = 1 - k,..., N - 1. \tag{5.26}$$

In addition, suppose $r \leqslant s - 1$ and that $2r \leqslant s + 1$ if $s < l$. Then (5.17) *and* (5.18) *hold with K replaced by*

$$K^* = \frac{k^{s-r+1}}{(s-1)!} \Gamma_{kr} \left[2^{s-1} + \sum_{j=s+1}^{l} (2\rho^*)^{j-s} \right],$$

where

$$\rho^* = \max_{1-k \leqslant i \leqslant N-1} \frac{|J_{ir}|}{\sigma_{is}}.$$

6. ERROR BOUNDS FOR A METHOD BASED ON LOCAL INTEGRALS

Fix integers $1 \leqslant l \leqslant k$, and suppose $\{\hat{p}_{ij}\}_{j=1}^{l}$ are the orthogonal polynomials ($\hat{p}_{ij} \in \mathscr{P}_j$) with respect to weight functions $\hat{w}_i$ defined on $[-1, 1]$, $i = 1 - k,..., N - 1$. Suppose $[\alpha_i, \beta_i] \subset [x_i, x_{i+k}]$, $i = 1 - k,..., N - 1$. Throughout this section we shall be interested in the B-spline approximation method

$$Q^I f(x) = \sum_{i=1-k}^{N-1} \sum_{j=1}^{l} \alpha_{ij}^I \lambda_{ij}^I f N_{i,k}(x), \tag{6.1}$$

where

$$\lambda_{ij}^I f = \int_{-1}^{1} \hat{w}_i(y) \hat{p}_{ij}(y) f\left(\frac{\beta_i - \alpha_i}{2} y + \frac{\alpha_i + \beta_i}{2}\right) dy \tag{6.2}$$

and α_{ij}^I are given by (3.9). (This is just Example 3.6 with $[\alpha_i, \beta_i]$, the support of the λ_{ij}^I, restricted to lie in $[x_i, x_{i+k}]$.)

In this section we state estimates for $|E_{r,s}^I|$ defined by (4.1) with Q replaced by Q^I. For $1 \leqslant m \leqslant N - 1$, let I_m be the smallest interval containing $[x_m, x_{m+1}]$ and $[\alpha_i, \beta_i]$, $i = m + 1 - k,..., m$.

THEOREM 6.1. *Let $1 \leqslant s \leqslant l \leqslant k$ and $1 \leqslant q \leqslant \infty$. If $f \in C^{s-1}[I_m]$, then for $0 \leqslant r < k$,*

$$\| E_{r,s}^I \|_{L_q[x_m, x_{m+1}]} \leqslant K \bar{\Delta}_m^{s-r-1+(1/q)} \omega(D^{s-1}f; \bar{\Delta}_m; I_m). \tag{6.3}$$

If $f \in L_p{}^s[I_m]$, $1 \leqslant p \leqslant \infty$, *then for* $0 \leqslant r < k$,

$$\| E^I_{r,s} \|_{L_q[x_m, x_{m+1}]} \leqslant K_m \bar{\Delta}_m^{s-r+(1/q)-(1/p)} \| D^s f \|_{L_p[I_m]} . \tag{6.4}$$

Here

$$K_m = \frac{k^{s+1} \Gamma_{kr}}{(s-1)!} \left(\frac{\bar{\Delta}_m}{\Delta_m} \right)^r \max_{m+1-k \leqslant i \leqslant m} \| w_i \|^{1/2}_{L_p[-1,1]} \sum_{j=1}^{l} \frac{k_{ij}}{(h_{ij})^{1/2}} (2\rho_m)^{j-1},$$

where

$$\rho_m = \max_{m+1-k \leqslant i \leqslant m} \frac{(x_{i+k} - x_i)}{(\beta_i - \alpha_i)}$$

and k_{ij} *and* h_{ij} *are the constants associated with the orthogonal polynomials* $\hat{p}_{ij}$.

Arguing as in Section 5 we easily obtain global estimates.

THEOREM 6.2. *Let* $1 \leqslant s \leqslant l \leqslant k$ *and* $1 \leqslant q \leqslant \infty$. *If* $f \in C^{s-1}[a, b]$, *then for* $0 \leqslant r < k$,

$$\| E^I_{r,s} \|_{L_q[a,b]} \leqslant K \bar{\Delta}^{s-r-1} \omega(D^{s-1} f; \bar{\Delta}; [a, b]). \tag{6.5}$$

If $f \in L_p{}^s[a, b]$, $1 \leqslant p \leqslant q$, *then for* $0 \leqslant r < k$,

$$\| E^I_{r,s} \|_{L_q[a,b]} \leqslant (2k-1) K \bar{\Delta}^{s-r+(1/q)-(1/p)} \| D^s f \|_{L_p[a,b]} . \tag{6.6}$$

Here

$$K = \frac{k^{s+1} \Gamma_{kr}}{(s-1)!} \left(\frac{\bar{\Delta}}{\Delta_{k-r}} \right)^r \max_{1-k \leqslant i \leqslant N-1} \| w_i \|^{1/2}_{L_1[\alpha_i, \beta_i]} \sum_{j=1}^{l} \frac{k_{ij}}{(h_{ij})^{1/2}} (2\rho)^{j-1},$$

with

$$\rho = \max_{1-k \leqslant i \leqslant N-1} \frac{(x_{i+k} - x_i)}{(\beta_i - \alpha_i)} .$$

Mesh independence results seem to be more difficult to obtain for Q^I than for Q^E. The difficulty is that in the estimates there are $j-1$ of the $(\beta_i - \alpha_i)$ factors in the denominator as well as r of the Δ factors. We content ourselves with only the simplest possible result.

THEOREM 6.3. *Suppose in Theorem* 6.2 *that for* $i = m+1-k, \ldots, m$,

$$[x_{i+1}, x_{i+k-1}] = [\alpha_i, \beta_i], \qquad r = 1 < s. \tag{6.7}$$

Then (6.3) *and* (6.4) *hold for* $r = 1 < s$ *with*

$$K = \frac{k^{s+1} \Gamma_{kr}}{(s-1)!} \max_{m+1-k \leqslant i \leqslant m} \sum_{j=1}^{l} \frac{k_{ij}}{(h_{ij})^{1/2}} (d_i - c_i)^{j-1} \| w_i \|^{1/2}_{L_1[\alpha_i, \beta_i]} .$$

7. PROJECTIONS

In this section we examine the question of when a B-spline approximation method Q of the form (3.1) is a *projection* onto $\mathscr{S}_{k,\pi}$. If $\mathscr{F}$ is the class of functions for which Q is defined, we will suppose $\mathscr{S}_{k,\pi} \subset \mathscr{F}$. The following lemma is well known, but we include its proof for completeness.

LEMMA 7.1. *Let $Q: \mathscr{F} \to \mathscr{S}_{k\pi}$ be the linear mapping defined by* (3.1) *for some set of linear functionals $\{\lambda_i\}_{1-k}^{N-1}$. Then Q is a projector* (*i.e.*, $Qs = s$ *for all* $s \in \mathscr{S}_{k\pi}$) *if and only if $\{\lambda_i\}_{1-k}^{N-1}$ is a dual basis to $\{N_{i,k}\}_{1-k}^{N-1}$; i.e.,*

$$\lambda_i N_{jk} = \delta_{ij}, \qquad i, j = 1-k,..., N-1. \tag{7.1}$$

Proof. Since $\{N_{jk}\}_{1-k}^{N-1}$ is a basis, Q is a projector if and only if $QN_{jk} = \sum_{i=1-k}^{N-1} (\lambda_i N_{jk}) N_{ik} = N_{jk}$, all $j = 1-k,..., N-1$. This is clearly equivalent to (7.1). ▌

Now we may ask when a dual basis $\{\lambda_i\}_{1-k}^{N-1}$ can be constructed from given sets $\{\lambda_{ij}\}_{j=1}^{k}$ of linear functionals, $i = 1-k,..., N-1$. We need $l = k$ since $\mathscr{P}_k \subset \mathscr{S}_{k,\pi}$ and so Q must reproduce polynomials of degree $k-1$. It would be natural to take $\lambda_i = \sum_{j=1}^{k} \alpha_{ij}\lambda_{ij}$ with $\{\alpha_{ij}\}_{j=1}^{k}$ given by (3.5) with $l = k$. In general, this is not sufficient to assure that Q is a projector (see Example 7.5 below). The following result (suggested to us by C. de Boor) gives a sufficient condition.

THEOREM 7.2. *For $i = 1-k,..., N-1$ let $\{\lambda_{ij}\}_{j=1}^{k}$ satisfy* (3.4), *and suppose $\{\lambda_{ij}\}_{j=1}^{k}$ all have support in one subinterval $[x_{\nu_i}, x_{\nu_i+1}]$ of $[x_i, x_{i+k}]$. Then with $\{\alpha_{ij}\}_{j=1}^{k}$ given by* (3.5), *the set $\{\lambda_i\}_{1-k}^{N-1}$ is a dual set to $\{N_{ik}\}_{1-k}^{N-1}$.*

Proof. Fix $1-k \leqslant i \leqslant N-1$. By (2.3), $\{N_{\mu k}\}_{\mu=\nu_i-k+1}^{\nu_i}$ is linearly independent over $[x_{\nu_i}, x_{\nu_i+1}]$, and hence span $\mathscr{P}_k$ in this interval. But then by (3.4) the determinant in the system

$$\lambda_i N_{\mu k} = \sum_{j=1}^{k} \alpha_{ij}\lambda_{ij}N_{\mu k} = \delta_{i\mu}, \qquad \mu = \nu_i - k + 1,..., \nu_i$$

is nonzero, and we can solve it uniquely for $\{\alpha_{ij}\}_{j=1}^{k}$. Now

$$\sum_{j=1}^{k} \alpha_{ij}\lambda_{ij}N_{\mu k} = 0, \qquad \mu = 1-k,..., \nu_i - k, \nu_i + 1,..., N-1,$$

automatically by the support properties of the $\{\lambda_{ij}\}_{j=1}^{k}$. Now $\{\lambda_i\}_{1-k}^{N-1}$ is a dual basis, and by Lemma 7.1 the corresponding Q is a projector. But then (3.3) must hold so $\{\alpha_{ij}\}_{j=1}^{k}$ must in fact be a solution of (3.5). ▌

We now give several examples.

EXAMPLE 7.3. Suppose in Example 3.4 that $l = k$ and that for $i = 1 - k, \ldots, N - 1$ the $\{\tau_{ij}\}_{j=1}^{k}$ are chosen from intervals $[x_{\nu_i}, x_{\nu_i+1}] \subset [x_i, x_{i+k}]$. Suppose also that if some τ_{ij} is at a knot x_μ, then the multiplicity of the τ_{ij} does not exceed k minus the multiplicity of the knot x_μ. (Then λ_{ij} can be evaluated on any $s \in \mathscr{S}_{k\pi}$, and Q is defined on a class $\mathscr{F}$ containing $\mathscr{S}_{k\pi}$. In particular, if d is the maximum multiplicity of the τ_{ij}'s, then Q is defined on $C^{d-1}[a, b]$ at least). Theorem 7.2 now assures that Q defined by (3.8) is a projector of $C^{d-1}[a, b]$ onto $\mathscr{S}_{k\pi}$. This example includes several projectors constructed in de Boor [4].

EXAMPLE 7.4. Suppose in Example 3.6 we take $l = k$ and insist that $[\alpha_i, \beta_i] \subset [x_{\nu_i}, x_{\nu_i+1}] \subset [x_i, x_{i+k}]$ for some ν_i, $i = 1 - k, \ldots, N - 1$. Then Q given by (3.16) is a projector of $L_1[a, b]$ onto $\mathscr{S}_{k,\pi}$.

EXAMPLE 7.5. Let $k = 2$ and $x_i = i$, $i = 1 - k, \ldots, N - 1$. Then

$$N_{i2}(x) = \begin{cases} x - i, & i \leqslant x \leqslant i + 1, \\ i + 2 - x, & i + 1 \leqslant x \leqslant i + 2, \\ 0, & \text{otherwise}. \end{cases}$$

Let $\lambda_{i1} = e_{\tau_{i1}}$ (evaluation at τ_{i1}) and $\lambda_{i2} = e_{\tau_{i2}}$, where $\tau_{i1} = i + \frac{1}{2}$, $\tau_{i2} = i + \frac{3}{2}$. Then if we seek a dual basis of the form $\lambda_i = \alpha_{i1}\lambda_{i1} + \alpha_{i2}\lambda_{i2}$, we will need, e.g. with $i = 0$,

$$\begin{pmatrix} \lambda_0 N_{-12} \\ \lambda_0 N_{02} \\ \lambda_0 N_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \alpha_{01} \\ \alpha_{02} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

But this has no solution. Thus no dual basis $\{\lambda_i\}_{1-k}^{N-1}$ can be constructed from the given $\{\lambda_{ij}\}$. This example shows that in Example 7.3 the requirement that the $\{\tau_{ij}\}_1^k$ lie in one subinterval of $[x_i, x_{i+k}]$ for each $i = 1 - k, \ldots, N - 1$ cannot be summarily dispensed with.

8. MULTIVARIATE APPROXIMATION METHODS

In this section we consider B-spline approximation methods for functions defined on a region Ω in $\mathbb{R}^2$. (The methods and results extend immediately to higher dimensions.)

Given $\Omega \subset \mathbb{R}^2$, let $H = [a, b] \times [\tilde{a}, \tilde{b}]$ be a rectangle with $\Omega \subset H$. Suppose $\pi = \{a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_N = b\}$ and $\tilde{\pi} = \{\tilde{a} = \tilde{x}_0 \leqslant \tilde{x}_1 \leqslant \cdots \leqslant \tilde{x}_{\tilde{N}} = \tilde{b}\}$

are partitions with multiplicities at most d and $\tilde{d}$, respectively. Let $\pi_e = \{x_i\}_{1-k}^{N-1+k}$ and $\tilde{\pi}_e = \{\tilde{x}_i\}_{1-\tilde{k}}^{\tilde{N}-1+\tilde{k}}$ be extensions as in Section 2. With $\{N_{ik}(x)\}_{i=1-k}^{N-1}$ and $\{\tilde{N}_{\tilde{i}\tilde{k}}(\tilde{x})\}_{\tilde{i}=1-\tilde{k}}^{\tilde{N}-1}$ univariate B-splines constructed as in Section 2 we may define

$$N_{i\tilde{i}k\tilde{k}}(x, \tilde{x}) = N_{ik}(x)\, \tilde{N}_{\tilde{i}\tilde{k}}(\tilde{x}),$$

for $i = 1-k,..., N-1$ and $\tilde{i} = 1-\tilde{k},..., \tilde{N}-1$. This is a collection of bivariate B-splines defined on $[x_{1-k}, x_{N+k-1}] \times [\tilde{x}_{1-\tilde{k}}, \tilde{x}_{\tilde{N}+\tilde{k}-1}]$.

Let $H_{i\tilde{i}} = [x_i, x_{i+1}] \times [\tilde{x}_{\tilde{i}}, \tilde{x}_{\tilde{i}+1}]$ and $\mathscr{Q} = \{(i, \tilde{i})\colon \operatorname{supp} N_{i\tilde{i}k\tilde{k}} \cap \Omega \neq \phi,\ 1-k \leqslant i \leqslant N-1,\ 1-\tilde{k} \leqslant \tilde{i} \leqslant \tilde{N}-1\}$. Suppose $\mathscr{F}$ is a linear space of functions defined on Ω, and suppose $\{\theta_{i\tilde{i}}\}_{(i,\tilde{i})\in\mathscr{Q}}$ is a collection of linear functionals defined on $\mathscr{F}$. Then for any $f \in \mathscr{F}$ we may define a B-spline approximation by

$$Qf(x, \tilde{x}) = \sum_{(i,\tilde{i})\in\mathscr{Q}} \theta_{i\tilde{i}} f\, N_{i\tilde{i}k\tilde{k}}(x, \tilde{x}). \tag{8.1}$$

The simplest way to construct such formulas is to take the tensor product of two univariate schemes. But if we do that, then we will get a scheme which usually will require information about f outside of Ω (unless, for example, Ω is a rectangle itself). In order to obtain a method applicable to functions defined on Ω, we need to consider (possibly different) univariate schemes for each $0 \leqslant i \leqslant N-1$ and $0 \leqslant \tilde{i} \leqslant \tilde{N}-1$.

For $1-k \leqslant i \leqslant N-1$, $1-\tilde{k} \leqslant \tilde{i} \leqslant \tilde{N}-1$, let $\lambda_{i\tilde{i}}$ be a linear functional defined on functions of the variable x on $[a, b]$ and let $\tilde{\lambda}_{i\tilde{i}}$ be a linear functional defined on functions of the variable $\tilde{x}$ on $[\tilde{a}, \tilde{b}]$. Suppose

$$\text{support } \lambda_{i\tilde{i}}\tilde{\lambda}_{i\tilde{i}} = [\text{support } \lambda_{i\tilde{i}}] \times [\text{support } \tilde{\lambda}_{i\tilde{i}}] \subset \Omega \qquad \text{for} \quad (i, \tilde{i}) \in \mathscr{Q}. \tag{8.2}$$

Now with $\theta_{i\tilde{i}} = \lambda_{i\tilde{i}}\tilde{\lambda}_{i\tilde{i}}$ we have an operator Q_* defined on functions on H by

$$Q_* f(x, \tilde{x}) = \sum_{\substack{1-k\leqslant i\leqslant N-1 \\ 1-\tilde{k}\leqslant \tilde{i}\leqslant \tilde{N}-1}} \theta_{i\tilde{i}} f\, N_{i\tilde{i}k\tilde{k}}(x, \tilde{x}), \tag{8.3}$$

all $(x, \tilde{x}) \in H$. If we are interested only in $(x, \tilde{x}) \in \Omega$, then Q_* reduces to Q defined in (8.1).

Let $\mathscr{P}_l^{(2)}$ be the class of all polynomials in two variables of total degree less than l. The following result is easily proved.

THEOREM 8.1. *Let* $\theta_{i\tilde{i}} = \lambda_{i\tilde{i}}\tilde{\lambda}_{i\tilde{i}}$. *Then*

$$Q_* g(x, \tilde{x}) = g(x, \tilde{x}), \qquad (x, \tilde{x}) \in H, \qquad \text{all} \quad g \in \mathscr{P}_l^{(2)}, \tag{8.4}$$

if and only if

$$Q_1 g_1(x) := \sum_{i=1-k}^{N-1} \lambda_{i\tilde{i}}\, g_1 N_{ik}(x) = g_1(x), \qquad x \in [a, b], \qquad \text{all} \quad g_1 \in \mathscr{P}_l, \tag{8.5}$$

and

$$Q_2 g_2(\tilde{x}) := \sum_{\tilde{i}=1-\tilde{k}}^{\tilde{N}-1} \tilde{\lambda}_{i\tilde{i}}\, g_2 \tilde{N}_{\tilde{i}\tilde{k}}(\tilde{x}) = g_2(\tilde{x}), \qquad \tilde{x} \in [\tilde{a}, \tilde{b}], \qquad \text{all} \quad g_2 \in \mathscr{P}_l, \tag{8.6}$$

for all $0 \leqslant i \leqslant N-1$, $0 \leqslant \tilde{i} \leqslant \tilde{N}-1$.

Since for $(x, \tilde{x}) \in \Omega$, $Q_* f(x, \tilde{x}) = Qf(x, \tilde{x})$, (8.5) and (8.6) also imply

$$Qg(x, \tilde{x}) = g(x, \tilde{x}), \qquad (x, \tilde{x}) \in \Omega, \qquad \text{all} \quad g \in \mathscr{P}_l^{(2)}. \tag{8.7}$$

Thus if for each $0 \leqslant i \leqslant N-1$ and $0 \leqslant \tilde{i} \leqslant \tilde{N}-1$ the corresponding univariate schemes are constructed to reproduce polynomials, so will Q. For example, the methods discussed in Examples 3.4–3.6 lead immediately to multidimensional analogs which reproduce polynomials.

As we saw in Section 3, it is most convenient to construct univariate schemes which reproduce polynomials by choosing the linear functionals as linear combinations of other simpler functionals. Thus, for example, we might have

$$\lambda_{i\tilde{i}} = \sum_{j=1}^{l} \alpha_{i\tilde{i}j} \lambda_{i\tilde{i}j} \tag{8.8}$$

and

$$\tilde{\lambda}_{i\tilde{i}} = \sum_{\tilde{j}=1}^{l} \tilde{\alpha}_{i\tilde{i}\tilde{j}} \tilde{\lambda}_{i\tilde{i}\tilde{j}}\,. \tag{8.9}$$

Then Q is given by (8.1) with

$$\theta_{i\tilde{i}} = \sum_{\tilde{j},j=1}^{l} \alpha_{i\tilde{i}j} \tilde{\alpha}_{i\tilde{i}\tilde{j}} \lambda_{i\tilde{i}j} \tilde{\lambda}_{i\tilde{i}\tilde{j}}\,.$$

We also note that if $\lambda_{i\tilde{i}j}$ and $\tilde{\lambda}_{i\tilde{i}\tilde{j}}$ annihilate polynomials of degree $j-1$ and $\tilde{j}-1$, respectively, then it is easily seen that

$$\theta_{i\tilde{i}}^* = \sum_{\tilde{j}+j \leqslant l} \alpha_{i\tilde{i}j} \tilde{\alpha}_{i\tilde{i}\tilde{j}} \lambda_{i\tilde{i}j} \tilde{\lambda}_{i\tilde{i}\tilde{j}} \tag{8.10}$$

has the property $\theta_{i\tilde{i}}\, g = \theta_{i\tilde{i}}^*\, g$ for all $g \in \mathscr{P}_l$. Thus if Q defined by (8.1) with $\theta_{i\tilde{i}}$ reproduces polynomials $\mathscr{P}_l^{(2)}$, then so does Q^* defined by (8.1) with $\theta_{i\tilde{i}}^*$.

We close this section with two lemmas useful for obtaining error bounds. Given $1 \leqslant s \leqslant k$, let

$$E_{r\tilde{r}s}(t, \tilde{t}) = \begin{cases} D^{r,\tilde{r}}(f - Qf)(t, \tilde{t}), & 0 \leqslant r + \tilde{r} < s, \\ D^{r,\tilde{r}}Qf(t, \tilde{t}), & s \leqslant r + \tilde{r} < k. \end{cases} \tag{8.11}$$

Arguing just as in Section 4, we obtain

LEMMA 8.2. *Suppose $Qg = g$ for all $g \in \mathscr{P}_l^{(2)}$, where $0 < s < l \leqslant k$. Then for any $g^* \in \mathscr{P}_s^{(2)}$,*

$$E_{r\tilde{r}s}(t, \tilde{t}) = \begin{cases} D^{r,\tilde{r}}R(t, \tilde{t}) - D^{r,\tilde{r}}QR(t, \tilde{t}), & 0 \leqslant r + \tilde{r} < s, \\ D^{r,\tilde{r}}QR(t, \tilde{t}), & s \leqslant r + \tilde{r} < k, \end{cases} \tag{8.12}$$

where $R(x, \tilde{x}) = f(x, \tilde{x}) - g^*(x, \tilde{x})$.

Now suppose $\lambda_{i\tilde{i}}$ and $\tilde{\lambda}_{i\tilde{i}}$ are given by (8.8) and (8.9) with $\lambda_{i\tilde{i}j}$ and $\tilde{\lambda}_{i\tilde{i}\tilde{j}}$ satisfying (cf. (3.7))

$$\lambda_{i\tilde{i}j} p_{i\tilde{i}\nu} = \delta_{j\nu}, \qquad j, \nu = 1, 2, \ldots, l$$
$$\tilde{\lambda}_{i\tilde{i}\tilde{j}} \tilde{p}_{i\tilde{i}\tilde{\nu}} = \delta_{\tilde{j}\tilde{\nu}}, \qquad \tilde{j}, \tilde{\nu} = 1, 2, \ldots, l$$

for some polynomials $p_{i\tilde{i}\nu}(x) = c_{i\tilde{i}\nu}(x - z_{i\tilde{i}\nu 1}) \cdots (x - z_{i\tilde{i}\nu q_\nu})$ and $\tilde{p}_{i\tilde{i}\tilde{\nu}}(\tilde{x}) = \tilde{c}_{i\tilde{i}\tilde{\nu}}(\tilde{x} - \tilde{z}_{i\tilde{i}\tilde{\nu}1}) \cdots (\tilde{x} - \tilde{z}_{i\tilde{i}\tilde{\nu}\tilde{q}_{\tilde{\nu}}})$. Let

$$\mathscr{Q}_{m\tilde{m}} = \{(i, \tilde{i}) \colon m + 1 - k \leqslant i \leqslant m, \tilde{m} + 1 - k \leqslant \tilde{i} \leqslant \tilde{m}\}.$$

LEMMA 8.3. *For any R as in Lemma 8.2 and $(t, \tilde{t}) \in H_{m,\tilde{m}}$,*

$$| D^{r,\tilde{r}} QR(t, \tilde{t})| \leqslant \max_{(i,\tilde{i}) \in \mathscr{Q}_{m,\tilde{m}}} \sum_{j,\tilde{j}=1}^{l} | c_{i\tilde{i}j} | \, | \tilde{c}_{i\tilde{i}\tilde{j}} | \, | \lambda_{i\tilde{i}j} \tilde{\lambda}_{i\tilde{i}\tilde{j}} R \, | \, A_{i\tilde{i}j} A_{i\tilde{i}\tilde{j}}, \tag{8.13}$$

where

$$A_{i\tilde{i}j} = \max_{\substack{i+1 \leqslant \nu_1, \cdots, \nu_{q_j} \leqslant i+k-1 \\ \nu_1, \cdots, \nu_{q_j} \text{ distinct}}} \frac{| x_{\nu_1} - z_{i\tilde{i}j1} | \cdots | x_{\nu_{q_j}} - z_{i\tilde{i}jq_j} |}{\Delta_{i,\tilde{i},m,k-1} \cdots \Delta_{i,\tilde{i},m,k-r}}$$

($\tilde{A}_{i\tilde{i}\tilde{j}}$ is defined similarly), and where $\Delta_{i,\tilde{i},m,\nu}$ are defined analogously to the univariate case.

9. ERROR BOUNDS FOR $C^{s-1}(\Omega)$ FUNCTIONS

In this section we obtain bounds for $E_{r\tilde{r}s}$ defined by (8.11) with $Q = Q^E$ defined by (8.1) and with $\theta_{i\tilde{i}} = \theta_{i\tilde{i}}^*$ given by (8.10) with

$$\begin{aligned} \lambda_{i\tilde{i}j} f &= [\tau_{i\tilde{i}1}, \ldots, \tau_{i\tilde{i}j-1}] f, \\ \tilde{\lambda}_{i\tilde{i}\tilde{j}} f &= [\tilde{\tau}_{i\tilde{i}1}, \ldots, \tilde{\tau}_{i\tilde{i}\tilde{j}-1}] f. \end{aligned} \tag{9.1}$$

We suppose $k = \tilde{k}$, that (8.2) holds, and that $\Lambda_{i\tilde{\imath}} = \text{support}\, \lambda_{i\tilde{\imath}} \subset [x_i, x_{i+k}]$ and $\tilde{\Lambda}_{i\tilde{\imath}} = \text{support}\, \tilde{\lambda}_{i\tilde{\imath}} \subset [\tilde{x}_{\tilde{\imath}}, \tilde{x}_{\tilde{\imath}+k}]$. If d and $\tilde{d}$ are the maximum multiplicities of the τ's and $\tilde{\tau}$'s, respectively, then Q^E is defined for all $f \in C^{s-1}(\Omega)$ with $d + \tilde{d} - 1 \leqslant s$.

To obtain bounds on $E_{r\tilde{r}s}$, we shall use Lemmas 8.2 and 8.3, and compare f with its Taylor expansion. To assure that the bound depends only on values of f in Ω we shall suppose Ω is locally convex with respect to the partition $\pi \times \tilde{\pi}$ and the support sets $\Lambda_{i\tilde{\imath}} \times \tilde{\Lambda}_{i\tilde{\imath}}$ of the linear functionals defining Q. By this we mean the following. For fixed $(t, \tilde{t}) \in \Omega$, let $m, \tilde{m}$ be such that $(t, \tilde{t}) \in H_{m\tilde{m}}$. We recall that the B-splines $N_{i\tilde{\imath}}$ are nonzero at $(t, \tilde{t})$ for $(i, \tilde{\imath}) \in \mathscr{Q}_{m,\tilde{m}} = \{(i, \tilde{\imath}) \colon m + 1 - k \leqslant i \leqslant m,\ \tilde{m} + 1 - k \leqslant \tilde{\imath} \leqslant \tilde{m}\}$. Then we say Ω is *locally convex* (cf. de Boor and Fix [6]) provided that for every $(t, \tilde{t}) \in \Omega$, for every $(\zeta, \tilde{\zeta}) \in \Lambda_{i\tilde{\imath}} \times \tilde{\Lambda}_{i\tilde{\imath}}$, and for every $(i, \tilde{\imath}) \in \mathscr{Q}_{m\tilde{m}}$, the line from $(t, \tilde{t})$ to $(\zeta, \tilde{\zeta})$ lies entirely in Ω.

We note that convex regions Ω are trivially locally convex with respect to any partition and any choice of linear functionals with supports in Ω. Polygonal regions with sides parallel to the coordinate axes are also locally convex provided the supports of the linear functionals are carefully chosen and provided the mesh is sufficiently fine.

Given $0 \leqslant m \leqslant N - 1$ and $0 \leqslant \tilde{m} \leqslant \tilde{N} - 1$, let $U_{m\tilde{m}} = \bigcup_{i,\tilde{\imath} \in \mathscr{Q}_{m\tilde{m}}}$ (convex hull $(\Lambda_{i\tilde{\imath}} \times \tilde{\Lambda}_{i\tilde{\imath}} \cup H_{m\tilde{m}})$). By the assumption on Ω, clearly $U_{m\tilde{m}} \subset \Omega$. Now for any $f \in C^{s-1}(U_{m\tilde{m}})$, and fixed $(t, \tilde{t}) \in H_{m\tilde{m}}$, we define

$$R_{(t,\tilde{t})}(x, \tilde{x}) = f(x, \tilde{x}) - \sum_{j=0}^{s-1} \sum_{\tilde{j}=0}^{s-j-1} \frac{D^{j,\tilde{j}} f(t, \tilde{t})(x - t)^j (\tilde{x} - \tilde{t})^{\tilde{j}}}{j!\,\tilde{j}!}.$$

We note that for any $(\zeta, \tilde{\zeta}) \in U_{m\tilde{m}}$ and $1 \leqslant j, \tilde{j} \leqslant s - 1$, $j + \tilde{j} \leqslant s + 1$,

$$D^{j-1,\tilde{j}-1} R_{(t,\tilde{t})}(x, \tilde{x}) = \sum_{\mu=0}^{s-j-\tilde{j}+1} \frac{(x - t)^\mu (\tilde{x} - \tilde{t})^{s-j-\tilde{j}-\mu+1} D^{j-1+\mu,\, s-j-\mu} R(\zeta, \tilde{\zeta})}{\mu!\,(s - j - \tilde{j} - \mu + 1)!}, \tag{9.2}$$

where $(\zeta, \tilde{\zeta})$ is on the line from $(t, \tilde{t})$ to $(x, \tilde{x})$. Also,

$$D^{j-1+\mu,\, s-j-\mu} R(\zeta, \tilde{\zeta}) = D^{j-1+\mu,\, s-j-\mu} f(\zeta, \tilde{\zeta}) - D^{j-1+\mu,\, s-j-\mu} f(t, \tilde{t}), \tag{9.3}$$

$\mu = 0, 1, \ldots, s - j - \tilde{j} + 1$. We define for any $\Delta > 0$ and any region Θ,

$$\omega(D^{s-1}\varphi; \Delta, \Theta) = \max_{0 \leqslant \nu \leqslant 0-1} \omega(D^{\nu, s-\nu-1}\varphi; \Delta; \Theta),$$

where

$$\omega(\psi; \Delta; \Theta) = \sup_{\substack{|\theta|, |\tilde{\theta}| \leqslant \Delta \\ (x,\tilde{x}), (x+\theta, \tilde{x}+\tilde{\theta}) \in \Theta}} |\psi(x + \theta, \tilde{x} + \tilde{\theta}) - \psi(x, \tilde{x})|.$$

Let $\Delta_{m\tilde{m}} = \bar{\Delta}_m + \bar{\tilde{\Delta}}_{\tilde{m}}$.

LEMMA 9.1. *With R as above for* $j, \tilde{j} = 1, 2, \ldots, l$,

$$| \lambda_{iij}\tilde{\lambda}_{ii\tilde{j}}R | \leqslant \frac{(\zeta_{iij} + \tilde{\zeta}_{ii\tilde{j}} - t - \tilde{t})^{s-j-\tilde{j}+1}\, k\omega(D^{s-1}f;\, \Delta_{m\tilde{m}}\, ;\, U_{m\tilde{m}})}{(s - j - \tilde{j} + 1)!\,(j-1)!\,(\tilde{j}-1)!}, \tag{9.4}$$

for $j + \tilde{j} \leqslant s + 1$, *where* $\zeta_{iij} \in \text{support}\, \lambda_{iij}$, $\tilde{\zeta}_{ii\tilde{j}} \in \text{support}\, \tilde{\lambda}_{ii\tilde{j}}$. *Moreover,*

$$| \lambda_{iij}\tilde{\lambda}_{ii\tilde{j}}R | \leqslant \frac{2^{j+\tilde{j}-s-1}\, k\omega(D^{s-1}f;\, \Delta_{m\tilde{m}}\, ;\, U_{m\tilde{m}})}{\sigma_{iij-1} \cdots \sigma_{ii\mu}\tilde{\sigma}_{ii\tilde{j}-1} \cdots \tilde{\sigma}_{iis-\mu+1}(\mu-1)!\,(s-\mu)!}, \tag{9.5}$$

for $j + \tilde{j} > s + 1$, *where* μ *is any integer with* $1 \leqslant \mu \leqslant j$ *and* $1 \leqslant s - \mu + 1 \leqslant \tilde{j}$.

Proof. For $j + \tilde{j} \leqslant s + 1$, we use (9.2) and (9.3) to obtain

$$| \lambda_{iij}\tilde{\lambda}_{ii\tilde{j}}R | = \frac{| D^{j-1,\tilde{j}-1}R(\zeta_{iij}, \tilde{\zeta}_{ii\tilde{j}})|}{(j-1)!\,(\tilde{j}-1)!} \leqslant k\omega(D^{s-1}f;\, \Delta_{m\tilde{m}}\, ;\, U_{m\tilde{m}}).$$

$$\times \sum_{\mu=0}^{s-j-\tilde{j}+1} \frac{(\zeta_{iij} - t)^{\mu}\,(\tilde{\zeta}_{ii\tilde{j}} - \tilde{t})^{s-j-\tilde{j}-\mu+1}}{\mu!\,(s-j-\tilde{j}-\mu+1)!\,(j-1)!\,(\tilde{j}-1)!}.$$

This leads immediately to (9.4).

For $j + \tilde{j} > s + 1$ and $1 \leqslant \mu \leqslant j$, $1 \leqslant s - \mu + 1 \leqslant \tilde{j}$, we use Lemma 2.2. Then

$$| \lambda_{iij}\tilde{\lambda}_{ii\tilde{j}}R | = |[\tau_{ii1}, \ldots, \tau_{iij}\, ;\, \tilde{\tau}_{ii1}, \ldots, \tilde{\tau}_{ii\tilde{j}}]R | \leqslant \sum_{\nu=0}^{j-\mu} \sum_{\tilde{\nu}=0}^{\tilde{j}-1-s+\mu}$$

$$\frac{|[\tau_{ii\nu-1}, \ldots, \tau_{ii\nu+\mu}\, ;\, \tilde{\tau}_{ii\tilde{\nu}+1}, \ldots, \tilde{\tau}_{ii\tilde{\nu}+1+s-\mu}]R | \binom{j-\mu}{\nu}\binom{\tilde{j}-1-s+\mu}{\tilde{\nu}}}{\sigma_{iij-1} \cdots \sigma_{ii\mu}\tilde{\sigma}_{ii\tilde{j}-1} \cdots \tilde{\sigma}_{iis-\mu+1}}. \tag{9.6}$$

Now as in the first part of the proof, each of the divided differences in the sum is bounded by

$$k\omega(D^{\mu-1,s-\mu}f;\, \Delta_{m\tilde{m}}U_{m\tilde{m}})/(\mu-1)!\,(s-\mu)!.$$

Now (9.5) follows easily. ∎

Using Lemmas 8.3 and 9.1 we obtain (cf. the proof of Theorem 5.2)

THEOREM 9.2. *Let* $d + \tilde{d} - 1 \leqslant s \leqslant l \leqslant k$ *and* $1 \leqslant q \leqslant \infty$. *If* $f \in C^{s-1}(U_{m\tilde{m}})$, *then for* $0 \leqslant r, \tilde{r} < k$,

$$\| E^{E}_{r\tilde{r}s} \|_{L_q[H_{m\tilde{m}} \cap \Omega]} \leqslant K_{m\tilde{m}}\Delta_{m\tilde{m}}^{s-r-\tilde{r}+(2/q)-1}\, \omega(D^{s-1}f;\, \Delta_{m\tilde{m}}\, ;\, U_{m\tilde{m}}), \tag{9.7}$$

where $K_{m\tilde{m}}$ is a constant depending on k, m, $\tilde{m}$, r, $\tilde{r}$, s, l, q and $\Delta_{m\tilde{m}}/\mathbf{\Delta}_{m,k-r}$, $\Delta_{m\tilde{m}}/\tilde{\mathbf{\Delta}}_{\tilde{m}k-\tilde{r}}$, ρ_m and $\tilde{\rho}_m$, with

$$\rho_m = \max_{(i,\tilde{i})\in\mathscr{Q}_{m\tilde{m}}} \frac{(x_{i+k} - x_i)}{\sigma_{i\tilde{i}s}}, \qquad \tilde{\rho}_{\tilde{m}} = \max_{(i,\tilde{i})\in\mathscr{Q}_{m\tilde{m}}} \frac{(\tilde{x}_{\tilde{i}+\tilde{k}} - \tilde{x}_{\tilde{i}})}{\tilde{\sigma}_{i\tilde{i}s}}.$$

We give two mesh independence results. First we consider $s = l$.

COROLLARY 9.3. *Suppose that in addition to the hypotheses of Theorem 9.2, we have*

$$\text{support } \lambda_{ii}\tilde{\lambda}_{\tilde{i}\tilde{i}} \subset J_{ir} \times \tilde{J}_{\tilde{i}\tilde{r}} \cap \Omega, \tag{9.8}$$

$(i, \tilde{i}) \in \mathscr{Q}_{m\tilde{m}}$, where J_{ir} is defined in (5.19) and $\tilde{J}_{\tilde{i}\tilde{r}}$ is defined similarly. Suppose also that $s = l$. Then (9.7) holds for $r + \tilde{r} \leqslant s - 1$ with a constant $K_{m\tilde{m}}$ depending on k, m, $\tilde{m}$, r, $\tilde{r}$, l, q and on

$$\rho^*_{m\tilde{m}} = \max_{(i,\tilde{i})\in\mathscr{Q}_{m\tilde{m}}} \frac{(\mathbf{\Delta}_{i,\tilde{i},m,k-r} + \mathbf{\Delta}_{i,\tilde{i},\tilde{m},k-\tilde{r}})}{\mathbf{\Delta}_{i,\tilde{i},\tilde{m},k-\tilde{r}}}, \tag{9.9}$$

$$\tilde{\rho}^*_{m\tilde{m}} = \max_{(i,\tilde{i})\in\mathscr{Q}_{m\tilde{m}}} \frac{(\mathbf{\Delta}_{i,\tilde{i},m,k-r} + \tilde{\mathbf{\Delta}}_{i,\tilde{i},\tilde{m},k-\tilde{r}})}{\tilde{\mathbf{\Delta}}_{i,\tilde{i},\tilde{m},k-\tilde{r}}}. \tag{9.10}$$

(We emphasize that in this case $K_{m\tilde{m}}$ does *not* depend on $\Delta_{m\tilde{m}}/\tilde{\mathbf{\Delta}}_{\tilde{m},k-\tilde{r}}$ or the ρ_m, $\tilde{\rho}_{\tilde{m}}$ in Theorem 9.2.)

Proof. By (9.8), the factors $\zeta_{iij} + \tilde{\zeta}_{\tilde{i}\tilde{i}\tilde{j}} - t - \tilde{t}$ in (9.4) are bounded by $\mathbf{\Delta}_{i,\tilde{i},\tilde{m},k-r} + \mathbf{\Delta}_{i,\tilde{i},\tilde{m},k-\tilde{r}}$. Thus these $s - j - \tilde{j} + 1$ factors can be used to cancel the shorter $\mathbf{\Delta}$'s in the denominator of (8.13). Then Lemma 5.4 can be applied. ∎

For $l < s$ we have

COROLLARY 9.4. *Suppose, in addition to the hypotheses of Theorem 9.2, that (9.8) holds. Suppose also that $r + \tilde{r} \leqslant s - 1$ and $2(r + \tilde{r}) \leqslant s + 1$. Suppose*

$$\rho^{**}_{m\tilde{m}} = \max_{(i,\tilde{i})\in\mathscr{Q}_{m\tilde{m}}} \frac{|J_{ir}|}{\sigma_{i\tilde{i}\omega}} < \infty \tag{9.11}$$

and

$$\tilde{\rho}^{**}_{m\tilde{m}} = \max_{(i,\tilde{i})\in\mathscr{Q}_{m\tilde{m}}} \frac{|J_{ir}|}{\tilde{\sigma}_{i\tilde{i}\tilde{\omega}}} < \infty, \tag{9.12}$$

where

$$\omega = \min([(s+1)/2], s - 2\tilde{r} + 2), \qquad \tilde{\omega} = \min([(s+1)/2], s - 2r + 2).$$

Then (9.7) holds with $K_{m\tilde{m}}$ depending only on k, m, $\tilde{m}$, r, $\tilde{r}$, l, q, s, $\rho^{**}_{m\tilde{m}}$, $\tilde{\rho}^{**}_{m\tilde{m}}$ and on the constants $\rho^*_{m\tilde{m}}$, $\tilde{\rho}^*_{m\tilde{m}}$ in (9.8) and (9.9).

(We note that the constants in (9.11)–(9.12) are bounded by $(k-r)(l-1)/\omega$ and $(k-\tilde{r})(l-1)/\tilde{\omega}$, respectively, if the τ's and $\tilde{\tau}$'s are taken equally spaced in J_{ir} and $\tilde{J}_{i\tilde{r}}$. A sufficient condition for J_{ir} and $\tilde{J}_{i\tilde{r}}$ to be nontrivial is $2r < k$ and $2\tilde{r} < k$.)

Proof. We need to consider the terms in (8.13) with $j + \tilde{j} \geqslant s + 2$ since the terms with $j + \tilde{j} \leqslant s + 1$ were estimated in Corollary 9.3. By (9.7) and the same argument as in the proof of Lemma 5.4, there are at least $j - 2r + 1$ $|x - z|$'s of length at most $|J_{ir}|$ and at least $\tilde{j} - 2\tilde{r} + 1$ $|\tilde{x} - \tilde{z}|$'s of length at most $|\tilde{J}_{i\tilde{r}}|$. We call these good factors. Now we claim that we can choose μ in (9.6) with $\mu \geqslant \omega$ and $s - \mu + 1 \geqslant \tilde{\omega}$ so that good factors can be used (via (9.11)–(9.12)) to cancel the σ's and $\tilde{\sigma}$'s in (9.6).

We recall (9.6) is obtained using Lemma 2.2. To explain the process of reducing

$$|[\tau_1, ..., \tau_j; \tilde{\tau}_1, ..., \tilde{\tau}_{\tilde{j}}]| \tag{9.13}$$

to a divided difference involving $s + 1$ points we write the following algorithm.

(1) If $j + \tilde{j} = s + 1$, then choose $\mu = j$ and exit;

(2) if $j \leqslant \tilde{j}$, go to (5);

(3) if $j - 2r + 1 \leqslant 0$, choose $\mu = j$ and exit;

(4) reduce the left side of (9.13) to one less point, cancelling a σ_{j-1} factor; go to (1);

(5) if $\tilde{j} - 2\tilde{r} + 1 \leqslant 0$, choose $\mu = s - 1 - \tilde{j}$ and exit;

(6) reduce the right side of (9.13) to one less point, canceling a $\tilde{\sigma}_{\tilde{j}-1}$ factor; go to (1).

We now show how the good factors can be used. If we exit from (3), then the factors $\tilde{\sigma}_{\tilde{j}-1}, ..., \tilde{\sigma}_{s-j+1}$ can be canceled by the $\tilde{j} - 2\tilde{r} + 1$ good $|\tilde{x} - \tilde{z}|$ factors since $2(r + \tilde{r}) \leqslant s + 1 \leqslant s + 2$. If we exit from (5), then the factors $\sigma_{j-1}, ..., \sigma_{s-1-\tilde{j}}$ can be canceled by the $j - 2r + 1$ good $|x - z|$ factors since $2(r + \tilde{r}) \leqslant s + 1$. Finally, if we exit from (1), then either j or $\tilde{j}$ equals $[(s + 1)/2]$, and the smallest σ or $\tilde{\sigma}$ factor canceled in (4) or (6) is at least $\sigma[(s + 1)/2]$ or $\tilde{\sigma}[(s + 1)/2]$, respectively.

Let $\Delta = \bar{\Delta} + \bar{\tilde{\Delta}}$.

THEOREM 9.5. *Let* $d + \tilde{d} - 1 \leqslant s \leqslant l \leqslant k$ *and* $1 \leqslant q \leqslant \infty$. *If* $f \in C^{s-1}(\Omega)$, *where* Ω *is locally convex in the sense defined above, then for* $0 \leqslant r, \tilde{r} < k$,

$$\| E_{r\tilde{r}s} \|_{L_q[\Omega]} \leqslant K\Delta^{s-r-\tilde{r}-1}\, \omega(D^{s-1}f; \Delta; \Omega),$$

where K is a constant depending on k, l, r, $\tilde{r}$, s, q and $\Delta/\boldsymbol{\Delta}_{k-r}$, $\Delta/\tilde{\boldsymbol{\Delta}}_{k-\tilde{r}}$, ρ and $\tilde{\rho}$, where

$$\rho = \max_{(i,\tilde{i})\in\mathscr{Q}} \frac{(x_{i+k} - x_i)}{\sigma_{i\tilde{i}s}}, \qquad \tilde{\rho} = \max_{(i,\tilde{i})\in\mathscr{Q}} \frac{(\tilde{x}_{\tilde{i}+k} - \tilde{x}_{\tilde{i}})}{\tilde{\sigma}_{i\tilde{i}s}}.$$

Clearly the analogs of the mesh independence results in Corollaries 9.3 and 9.4 hold for the global Theorem 9.5.

10. Error Bounds in Sobolev Spaces

In this section we assume Ω is a region in $\mathbb{R}^2$ which is locally convex in the sense defined in Section 9. We are concerned with approximating functions in the usual Sobolev space $W_p^s(\Omega)$, with norm given by

$$\|f\|_{W_p^s(\Omega)} = \left(\sum_{\nu=0}^{s} |f|_{\nu,p,\Omega}^p\right)^{(1/p)},$$

$$|f|_{\nu,p,\Omega} = \left(\sum_{\mu=0}^{\nu} \int_\Omega |D^{\mu,\nu-\mu} f|^p\right)^{(1/p)}.$$

We recall that $W_p^s(\Omega) \subset C^{v-1}(\Omega)$ for $v - 1 < s - 2/p$, (see [14, p. 69]).

We will approximate $f \in W_p^s(\Omega)$ by the B-spline method Q^E defined in Section 9. Thus in order to compute the $\lambda_{i\tilde{i}j}$ and $\tilde{\lambda}_{i\tilde{i}\tilde{j}}$ in (9.1), we need to assume that d and $\tilde{d}$, the maximum multiplicities of the τ and $\tilde{\tau}$'s in (9.1), are such that $d + \tilde{d} - 2 < v - (2/p)$.

We call a region $U \subset \mathbb{R}^2$ *starlike* if there exists a ball B such that for every $(x, \tilde{x}) \in U$ and every $(y, \tilde{y}) \in B$, the line between these points lies in U.

Lemma 10.1. *Let U be a starlike region. Suppose U is contained in a sphere of diameter Δ. Suppose $\varphi \in W_p^s(U)$, $1 < p < \infty$, $s \geq 1$. Then there exists a polynomial $s_\varphi \in \mathscr{P}_s^{(2)}$ (see [14, p. 55]) such that $R = \varphi - s_\varphi$ satisfies*

$$|D^{\alpha_1,\alpha_2} R(t, \tilde{t})| \leq K\Delta^{s-\alpha_1-\alpha_2-(2/p)} |\varphi|_{s,p,U}, \tag{10.1}$$

for $0 \leq \alpha_1 + \alpha_2 < s - 2/p$, where K is a constant independent of U and of φ. If $1 < q < \infty$, then

$$\|R\|_{W_q^j(U)} \leq K\Delta^{s-j+(2/q)-(2/p)} |\varphi|_{s,p,U}, \tag{10.2}$$

for $0 \leq j < s - (2/p)$. Moreover, (10.2) *also holds for j and q satisfying $s - (2/p) \leq j$ and $1 < q < 2p/[2 - (s - j)p]$.*

Proof. These results are essentially the theorem of Sobolev [14, p. 69]. For example, to prove (10.1), we have (assuming for the moment that $R \in C^s(\Omega)$) (see [14, p. 70]) that

$$D^{\alpha_1,\alpha_2} R(t, \tilde{t}) = \iint_U \frac{1}{r^{2-s+\alpha_1+\alpha_2}} \sum_{\nu=0}^{s} w_{\nu,s-\nu}^{\alpha_1,\alpha_2}(t, \tilde{t}; x, \tilde{x}) D^{\nu,s-\nu} R(x, \tilde{x})\, dx\, d\tilde{x},$$

where $r = [(x-t)^2 + (\tilde{x}-\tilde{t})^2]^{1/2}$, and $w_{\nu,s-\nu}^{\alpha_1,\alpha_2}$ is an appropriate bounded function. Then

$$\begin{aligned} &| D^{\alpha_1,\alpha_2} R(t, \tilde{t})| \\ &\leqslant \sup_{(x,\tilde{x})\in U} | w_{\nu,s-\nu}^{\alpha_1,\alpha_2}(t, \tilde{t}; x, \tilde{x})| \cdot \left[\iint_U r^{-(2-s+\alpha_1+\alpha_2)p'}\, dx\, d\tilde{x}\right]^{(1/p')} \cdot | R |_{s,p,U}, \end{aligned}$$

where $(1/p) + (1/p') = 1$. Now if U is contained in a sphere of diameter Δ, then

$$\begin{aligned} &\left(\iint_U r^{-(2-s+\alpha_1+\alpha_2)p'}\, dx\, d\tilde{x}\right)^{(1/p')} \\ &\quad\leqslant \left(\int_0^{2\pi} d\theta \int_0^{\Delta} \rho^{1-(2-s+\alpha_1+\alpha_2)p'}\, d\rho\right)^{(1/p')} \\ &\quad= \left(\frac{2\pi}{2-(2-s+\alpha_1+\alpha_2)p'}\right)^{(1/p')} \Delta^{s-\alpha_1-\alpha_2-(2/p)}. \end{aligned}$$

With some effort it can be seen that $| w_{\nu,\mu-1}(t, \tilde{t}, x, \tilde{x})| \leqslant \text{Con} < \infty$ for all $(t, \tilde{t})$, $(x, \tilde{x}) \in \mathbb{R}^2$ (with $\alpha_1 = \alpha_2 = 0$, Con $= 1$). This proves (10.1) since $| R |_{s,p,U} = | \varphi |_{s,p,U}$. The proof of (10.2) is similar. ∎

In the next theorem we apply Lemma 10.1 to the set $U_{m\tilde{m}}$ defined in Section 9. This is clearly a starlike region. Results similar to Lemma 10.1 (with indirect proofs) have been given for regions which are regular or strongly Lipschitz, but without precise knowledge of the constants (see, e.g., Jerome [11]; and references therein). We have followed Sobolev [14] because we wanted precise knowledge of how the constants depend on the region.

THEOREM 10.2. *Let* $1 < p < \infty$, $d + \tilde{d} - 2 < s - (2/p)$, *and* $s \leqslant l$. *Suppose* $f \in W_p^s(U_{m\tilde{m}})$. *Then*

$$\| E_{r\tilde{r}s}^{E} \|_{L_q(H_{m\tilde{m}} \cap \Omega)} \leqslant K_{m\tilde{m}} \Delta^{s-r-\tilde{r}+(2/q)-(2/p)} \, | f |_{s,p,U_{m\tilde{m}}} \tag{10.3}$$

for $1 \leqslant q \leqslant \infty$ *if* $0 \leqslant r + \tilde{r} < s-(2/p)$, *and for* $1 < q < 2p/[2-(s-r-\tilde{r})p]$ *if* $s - 2/p \leqslant r + \tilde{r}$.

The constant $K_{m\tilde{m}}$ depends on the same parameters as in Theorem 9.2.

Proof. We use Lemmas 8.2, 8.3, and 10.1 with $R = f - g$ and $g \in P_l^{(2)}$, the polynomial in Lemma 10.1. Now $\| D^{r,\tilde{r}} R \|_{L_q[H_{m,\tilde{m}} \cap \Omega]}$ is bounded using (10.1) or (10.2). For $D^{r\tilde{r}}QR$ we use (8.13). Now for $j + \tilde{j} \leqslant d + \tilde{d}$, using (10.1), we have

$$| \lambda_{iij} \tilde{\lambda}_{i\tilde{i}\tilde{j}} R | = \frac{| D^{j-1,\tilde{j}-1} R(\zeta_{iij}, \tilde{\zeta}_{i\tilde{i}\tilde{j}})|}{(j-1)!\,(\tilde{j}-1)!} \leqslant \text{Con } \Delta_{m\tilde{m}}^{s-j-\tilde{j}+2-(2/p)} \, | f |_{s,p,U_{m\tilde{m}}}.$$

For $j + \tilde{j} > d + \tilde{d}$, using Lemma 2.2 as in the proof of Lemma 9.1, we can reduce the $(j, \tilde{j})$ divided difference to a sum of $(\mu, d + \tilde{d} - \mu)$ divided differences with $1 \leqslant \mu \leqslant d, j$ and $1 \leqslant d + \tilde{d} - \mu - 1 \leqslant \tilde{j}$:

$$| \lambda_{iij} \tilde{\lambda}_{i\tilde{i}\tilde{j}} R | \leqslant \sum_{\nu=0}^{j-\mu} \sum_{\tilde{\nu}=0}^{\tilde{j}+\mu-d-\tilde{d}-2}$$

$$\times \frac{\binom{j-\mu}{\nu}\binom{\tilde{j}+\mu-d-\tilde{d}-2}{\tilde{\nu}} \, |[\tau_{ii\nu+1}, \ldots, \tau_{ii\nu+\mu} \,;\, \tilde{\tau}_{i\tilde{i}\tilde{\nu}+1}, \ldots, \tilde{\tau}_{i\tilde{i}\tilde{\nu}+d+\tilde{d}-\mu}] R \,|}{\sigma_{iij-1} \cdots \sigma_{ii\mu} \tilde{\sigma}_{i\tilde{i}\tilde{j}-1} \cdots \tilde{\sigma}_{i,\tilde{i},d+\tilde{d}-\mu}}.$$

Now the divided differences can again be estimated by (10.1), and the result follows. ∎

COROLLARY 10.3. *Suppose in Theorem* 10.2 *that we also have* (9.8) *and* $r + \tilde{r} \leqslant s - 1$ *and* $2(r + \tilde{r}) \leqslant \nu + 1$. *Suppose* (9.11) *and* (9.12) *hold with* $\omega = \min([(\nu + 1)/2], \nu - 2\tilde{r} + 2)$ *and* $\tilde{\omega} = \min([(\nu + 1)/2], \nu - 2r + 2)$. *Then* (10.3) *holds with $K_{m\tilde{m}}$ depending on k, m, $\tilde{m}$, r, $\tilde{r}$, l, q, s and the constants* $\rho_{m\tilde{m}}^{**}$, $\tilde{\rho}_{m\tilde{m}}^{**}$ *in* (9.11)–(9.12) *and* $\rho_{m\tilde{m}}^{*}$, $\tilde{\rho}_{m\tilde{m}}^{*}$ *in* (9.8) *and* (9.9).

Using Jensen's inequality, we immediately obtain

THEOREM 10.4. *Let* $1 < p < \infty$, $d + \tilde{d} - 2 \leqslant s - 1 < s - (2/p)$, *and* $s \leqslant l$. *Suppose* $f \in W_p^s(\Omega)$. *Then*

$$\| E_{r\tilde{r}s}^{E} \|_{L_q(\Omega)} \leqslant K \Delta^{s-r-\tilde{r}+(2/q)-(2/p)} \, | f |_{s,p,\Omega},$$

for $p \leqslant q \leqslant \infty$ *if* $0 \leqslant r + \tilde{r} < s - (2/p)$ *and for* $p \leqslant q < 2p/[2 - (s - r - \tilde{r})p]$ *if* $s - (2/p) \leqslant r + \tilde{r}$. *The constant K depends on the same parameters as in Theorem* 9.3.

The analog of Corollary 10.3 also holds for mesh independence in this global estimate.

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